Math 280 Solutions for November 6

Pythagoras Level

1. (Missouri 1997 Session 1, #3) Expansion of the sines and cosines gives

$$f(x,y) = (a+c)\sin x\cos y + (a-c)\cos x\sin y + (b+d)\cos x\cos y + (d-b)\sin x\sin y,$$

So, if the factorization is possible,

$$f(x,y) = (A\sin x + B\cos x)(C\sin y + D\cos y),$$

or

$$f(x,y) = AD\sin x\cos y + BC\cos x\sin y + BD\cos x\cos y + AC\sin x\sin y.$$

Thus we need to have

$$\begin{aligned} a+c &= AD \\ a-c &= BC \\ b+d &= BD \\ d-b &= AC. \end{aligned}$$

Hence,

$$\frac{a+c}{d-b} = \frac{AD}{AC} = \frac{D}{C} = \frac{BD}{BC} = \frac{b+d}{a-c}.$$

Therefore,

$$a^2+b^2 = c^2+d^2.$$

2. (Missouri 1997 Session 2, #1) No a_i can be greater than 4 since one could increase the product by replacing 5 by $2\cdot 3$, 6 by $3\cdot 3$, 7 by $3\cdot 4$, etc. There cannot be both a 2 and a 4 or three 2's among the a_i since $2\cdot 4 < 3\cdot 3$ and $2\cdot 2\cdot 2 < 3\cdot 3$. Also, there cannot be two 4's since $4\cdot 4 < 2\cdot 3\cdot 3$. Clearly, no a_i is a 1. Hence the a_i are 3's except possibly for a 4 or for a 2 or for two 2's. Since $2012 = 3\cdot 670 + 2$, the only exception is a 2 and n = 670.

Newton Level

3. (Missouri 1998 Session 1, #3) Integration by parts once yields

$$\int_a^b \frac{(b-x)^m}{m!}\frac{(x-a)^n}{n!}dx = \int_a^b \frac{(b-x)^{m-1}}{(m-1)!}\frac{(x-a)^{n+1}}{(n+1)!}dx$$

So continuing we get

$$\int_a^b \frac{(b-x)^m}{m!}\frac{(x-a)^n}{n!}dx = \int_a^b \frac{(x-a)^{n+m}}{(n+m)!}dx = \frac{(b-a)^{n+m+1}}{(n+m+1)!}.$$

Now

$$\int_0^1 (1-x^2)^n dx = \frac{1}{2}\int_{-1}^1 (1-x)^n(x-(-1))^n dx = \frac{1}{2}(n!)^2\frac{2^{2n+1}}{(2n+1)!} = \frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n+1)}.$$

4. (Missouri 1998 Session 2, #3) If S is the desired sum, then

$$2S = \sum_{i=1}^{\infty} \left(\frac{1}{(6i-1)^2} + \frac{1}{(6i+1)^2} \right) = \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots .$$

Now using the hint we obtain

$$\begin{aligned} \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots &= \frac{\pi^2}{24}, \\ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots &= \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}, \\ \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \cdots &= \frac{\pi^2}{72} \end{aligned}$$

So

$$2S + 1 = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots = \frac{\pi^2}{8} - \frac{\pi^2}{72} = \frac{\pi^2}{9}$$

Thus

$$S = \frac{\pi^2 - 9}{18}.$$

Wiles Level

5. (IMC 2009 Day 1 #2) A straightforward calculation shows that $(A-B)C = BA^{-1}$ is equivalent to $AC - BC - BA^{-1} + AA^{-1} = I$, where I denotes the identity matrix. This is equivalent to $(A-B)(C + A^{-1}) = I$. Hence, $(A-B)^{-1} = C + A^{-1}$, meaning that $(C + A^{-1})(A - B) = I$ also holds. Expansion yields the desired result.

6. (IMC 1994 Day 2 #3) Set $g(x) = (f(x) + f'(x) + \cdots + f^{(n)}(x))e^{-x}$. From the assumption one gets $g(a) = g(b)$. Then there exists $c \in (a, b)$ such that $g'(c) = 0$. Replacing in the last equality $g'(x) = (f(n+1)(x) - f(x))e^{-x}$ we finish the proof.