Math 280 Problems for September 4

Pythagoras Level

#1 Find the determinant of the $n \times n$ matrix $A = [a_{ij}]$ where

$$a_{ij} = \begin{cases} (-1)^{i-j} & \text{if } i \neq j, \\ 2 & \text{if } i = j. \end{cases}$$

#2 Consider a sequence of integers 1, 3, 2,-1, . . . , where each term is equal to the term preceding it minus the term before that. Whats the sum of the first 2009 terms?

Newton Level

#3 If $\lim_{x\to\infty} \left(\frac{x+2a}{x+a}\right)^x = 8$, what is a?

#4 The following figure consists of infinitely many squares and circles, with a circle inscribed in each square and a square inscribed in each circle. The outermost square has side length 1. Find the total shaded area.

Wiles Level

#5 Let $f(x)$ be a function that is continuously differentiable on $[0,1]$, with the following properties:

- $f(0) = 0$
- $f(1) = 1$
- $x < f(x) < 1$ for all $x \in (0,1)$.

Prove that for every positive integer n, there exist n distinct points $c_1, c_2, \ldots, c_n$ in $(0,1)$ such that

$$f'(c_1)f'(c_2)\cdots f'(c_n) = 1.$$

#6 Let $0 < a < b$. Evaluate

$$\lim_{p\to 0} \left(\int_0^1 (bx + a(1-x))^p dx\right)^{\frac{1}{p}}.$$