Math 280 Problems for September 11

Pythagoras Level

[New Jersey MAA 2009 Ind. #3] Write $n = a_1 b + a_0$ where $a_0, a_1 \in \{0, 1, \ldots, b-1\}$. Then $r_b(n) = a_0 b + a_1$, thus

$$n + r_b(n) = (a_1 + a_0)b + (a_0 + a_1) = (b+1)(a_0 + a_1).$$

So if $n + r_b(n)$ is a perfect square, $b+1$ must divide $a_0 + a_1$. Thus there are $b-2$ pairs of digits that work:

$$2(b-1),\ 3(b-2),\ \ldots,\ (b-1)2.$$

[Putnam 2008 A2] Barbara. If Alan puts a number in the ith row jth column and i is even, then Barbara puts the same number in the $(i-1)$th row jth column. If i is odd, then Barbara puts the same number in the $(i+1)$th row jth column. In this way at the end there will be (at least) two identical rows, hence the determinant will be zero.

Newton Level

[NJ MAA 2009 Ind. #7] Find the power series of $g(x)$ as follows:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots \\ g(x) = & e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + + \frac{x^8}{4!} + \frac{x^{10}}{5!} \cdots + \frac{x^n}{n!} + \cdots \end{aligned}$$

The coefficient on x^{2010} must be $g^{(2010)}(0)/2010!$, thus $g^{(2010)}(0) = 2010!/1005!$.

[NJ MAA 2007 #12] The sequence is increasing and bounded:
Increasing: $a_2 > a_1$ and the function $f(x) = \sqrt{2}^x$ is increasing, thus $a_{n+1} = \sqrt{2}^{a_n} > a_n$.
Bounded: Induction shows $a_n \leq 2$.
Thus the sequence converges. Its limit satisfies $L = \sqrt{2}^L$. This has solutions $L = 2$ and $L = 4$. Since $a_n \leq 2$, we find that the limit is 2.

Wiles Level

[Putnam 2008 A1] The function $g(x) = f(x, 0)$ works. Substituting $(x, y, z) = (0, 0, 0)$ into the given functional equation yields $f(0, 0) = 0$, whence substituting $(x, y, z) = (x, 0, 0)$ yields $f(x, 0) + f(0, x) = 0$. Finally, substituting $(x, y, z) = (x, y, 0)$ yields $f(x, y) = -f(y, 0) - f(0, x) = g(x) - g(y)$.

[NJ MAA 2009 Team #4] First note that there are 2^{n-1} numbers which have n digits, each of which is a 0 or 1. So modulo grouping of the terms, this series is "less than" the series,

$$\frac{1}{1} + 2\frac{1}{10} + 4\frac{1}{100} + 8\frac{1}{1000} + \cdots$$

This new series converges (to $1/(1 - 1/5) = 5/4$), as it is a geometric series with ratio $1/5$. Therefore, by the Comparison Test, the original series must also converge. If one is troubled by the grouping of terms, there are a variety of ways to make the justification completely rigorous. For example, note that each partial sum of the original series is less than a partial sum of the geometric series, which is in turn less than $5/4$. Thus, the sequence of partial sums of the original is a bounded sequence, and clearly monotone increasing as all terms in the series are non-negative. Therefore, the Monotone Convergence Theorem would imply convergence of the original series.