Math 280 Problems for September 18

Pythagoras Level

#1. (a) Show that the sum

$$1 + 2 + 3 + \cdots + 2009 + 2010 + 2009 + \cdots + 3 + 2 + 1$$

is the square of an integer.
(b) Generalize the result in (a), with proof.

#2. A game board consists of a linear path of 2010 squares, numbered from 1 to 2010. A game piece is initially on square 1, and two players alternately move it. On each move a player advances the piece by 1, 2, 3, 4, 5 or 6 squares. Thus, the first player advances the piece to square 2, 3, 4, 5, 6 or 7. The player who moves onto square 2010 wins. Describe a winning strategy for one of the players, and make clear that it wins.

Newton Level

#3. Suppose that the function f satisfies $f'(x) = 1 + f(x)$ for all x. If $f(2) = 3$, find:
(a) $f^{(10)}(2)$ (where $f^{(10)}$ denotes the 10th derivative of f);
(b) $f(3)$.
Justify your answers.

#4. Evaluate

$$\lim_{x\to\infty} \int_x^{2x} \frac{dt}{\sqrt{t^3+4}},$$

and justify your answer.

Wiles Level

#5. Prove that for every rational number a, the equation

$$y = \sqrt{x^2 + a}$$

has infinitely many solutions (x, y) with x and y rational.

#6.For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]