Math 280 Solutions for October 2

Pythagoras Level

1. (Ohio MAA 2006 #4) If
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
and
$$B = \begin{pmatrix} x & y \\ z & t \end{pmatrix},$$
then after a direct calculation we get
$$AB - BA = \begin{pmatrix} bz - cy & ay + bt - bx - dy \\ cx + dz - az - ct & cy - bz \end{pmatrix}.$$
The sum of the diagonal elements for $AB - BA$ is 0, while the sum of the diagonal elements for the desired matrix is 5. So no such A and B exist.

2. (Ohio MAA 2006 #2) Denote the given integers by $a_1, a_2, a_3 ... a_n$. Define:
$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ &\vdots \\ S_n &= a_1 + a_2 + \cdots + a_n \end{aligned}$$
If one of the numbers S_1, S_2, ... S_n is a multiple of n we are done. Otherwise all possible remainders upon division of these numbers by n are 1, 2, 3, ... $n-1$, i.e., we get more numbers than possible remainders. Therefore, among the numbers S_1, S_2, ... S_n there are two numbers, say S_k and S_{k+t} which give the same remainders upon division by n. We are done because $S_{k+t} - S_k = a_{k+1} + a_{k+2} + \cdots + a_{k+t}$ is a multiple of n.

Newton Level

3. (Ohio MAA 2006 #5) By using the (known) limit
$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2},$$
we can write
$$\lim_{x \to 0} \frac{1 - \cos(1 - \cos x)}{x^2} = \lim_{x \to 0} \frac{1 - \cos(1 - \cos x)}{(1 - \cos x)^2} \left(\frac{1 - \cos x}{x^2} \right)^2 = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$
So we can take $n = 2$ and $a = \frac{1}{8}$.

4. (Putnam 2008 B2) We claim that

$$F_n(x) = (\ln x - a_n)x^n/n!, \text{ where } a_n = \sum_{k=1}^{n} 1/k.$$

Indeed, temporarily write $G_n(x) = (\ln x - a_n)x^n/n!$ for $x > 0$ and $n \geq 1$. Then

$$\lim_{x \to 0} G_n(x) = 0$$

and

$$G_n'(x) = (\ln x - a_n + 1/n)x^{n-1}/(n-1)! = G_{n-1}(x)$$

and the claim follows by the Fundamental Theorem of Calculus and induction on n.

Given the claim, we have $F_n(1) = -a_n/n!$ and so we need to evaluate

$$-\lim_{n \to \infty} \frac{a_n}{\ln n}$$

But since the function $1/x$ is strictly decreasing for x positive,

$$\sum_{k=2}^{n} 1/k = a_n - 1$$

is bounded below by

$$\int_2^n dx/x = \ln n - \ln 2$$

and above by

$$\int_1^n dx/x = \ln n$$

It follows that

$$\lim_{n \to \infty} \frac{a_n}{\ln n} = 1$$

and the desired limit is -1.

Wiles Level

5. (Ohio MAA 2006 #1) Evaluating at $(0, 0)$ gives

$$f(0+0) = f(0)f(1) + f(0)f(1)$$

so that $f(1) = 1/2$ also. Then

$$f(x) = f(x+0) = f(x)f(1) + f(0)f(1-x)$$

and we find $f(x) = f(1-x)$. Substituting into the original equation shows: $f(x+y) = 2f(x)f(y)$. Then

$$f(x+1) = 2f(x)f(1) = f(x)$$

and we also get

$$f(-x) = f(1-(-x)) = f(1+x) = f(x).$$

Therefore

$$f(x-y) = 2f(x)f(-y) = 2f(x)f(y) = f(x+y).$$

Consequently,

$$f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = f\left(\frac{x}{2} - \frac{x}{2}\right) = f(0) = 1/2.$$

6. (Ohio MAA 2006 #3) If $g \in G$, then $gA \cap A \neq \emptyset$ since A has more than half as many elements as G. Therefore, there exist a_1 and a_2 such that $ga_1 = a_2$ and hence $g = a_2 a_1^{-1}$.