Math 280 Solutions for October 9

Pythagoras Level

1. The increasing sequence $S = \{2, 3, 5, 6, 7, 10, 11, \ldots\}$ consists of all positive integers which are neither a perfect square nor a perfect cube. What is the 500th term of S?

   [NES-MAA 2008 #4] Clearly the desired term n is unique and greater than 500. Since the squares and cubes thin out as we go, n is not likely to be much larger than 500.

   The number of squares less than 500 is 22 ($22^2 = 484$) and the number of cubes is 7 ($7^3 = 343$ but $8^3 = 512$). Thus, up to 500, no more than 29 numbers fail to be in the set S. In fact, the numbers 1 and 64 are both perfect squares and perfect cubes. Therefore, only 27 integers fail membership in S, making 500 the 473rd member of S. Counting forward 27 entries brings us to 528 since only 512 fails to be in S. The 500th term of S is 528.

2. Solve for x in terms of c:
$$2\log_x c - \log_{cx} c - 3\log_{c^2x} c = 0$$

   [NES-MAA 2008 #3] If $y = \log_a x$, then $a^y = x \Rightarrow a = x^{1/y} \Rightarrow \frac{1}{y} = \log_x a$. Ino ther words $\log_a x$ and $\log_x a$ are reciprocals. So rewrite the original equation as
$$\frac{2}{\log_c x} - \frac{1}{\log_c cx} - \frac{3}{\log_c c^2x} = 0.$$
   Set $k = \log_c x$. Then $\log_c cx = \log_c x + \log_c c = k + 1$ and $\log_c c^2x = \log_c x + 2\log_c c = k + 2$. So the equation becomes:
$$\frac{2}{k} - \frac{1}{k+1} - \frac{3}{k+2} = 0 \Longrightarrow \frac{-2k^2 + k + 4}{k(k+1)(k+2)} = 0$$
   Thus $-2k^2 + k + 4 = 0$. This has solutions $k = \frac{1\pm\sqrt{33}}{4}$. So $x = c^k = c^{(1\pm\sqrt{33})/4}$.

Newton Level

3. Alice, Bob, and Carol repeatedly take turns tossing a fair regular six-sided die. Alice begins; Bob always follows Alice; Carol always follows Bob; and Alice always follows Carol. Find the probability that Carol will be the first to toss a six.

   [NES-MAA 2008 #5] If Carol wins in the first round, then she must have rolled a six after two non-sixes have occurred. This happens with probability
$$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{6}\left(\frac{5}{6}\right)^2.$$
   If Carol wins in the second round, five non-sixes preceded her lucky six. This happens with probability
$$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{6}\left(\frac{5}{6}\right)^5.$$
   This pattern continues. Carol wins in the third round with probability $\frac{1}{6}\left(\frac{5}{6}\right)^8$ and in the fourth with probability$\frac{1}{6}\left(\frac{5}{6}\right)^1 1$. It is possible (though unlikely) that the game could continue forever. The probability that Carol wins the game is equal to the sum of the probabilities that she wins in any given round:
$$\begin{aligned}\frac{1}{6}\left(\frac{5}{6}\right)^2 + \frac{1}{6}\left(\frac{5}{6}\right)^5 + \frac{1}{6}\left(\frac{5}{6}\right)^8 + \cdots &= \frac{1}{6}\left(\frac{5}{6}\right)^2\left[1 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^6 + \cdots\right] \\ &= \frac{1}{6}\left(\frac{5}{6}\right)^2 \sum_{n=0}^{\infty}\left(\frac{5}{6}\right)^{3n} = \frac{1}{6}\left(\frac{5}{6}\right)^2 \sum_{n=0}^{\infty}\left(\frac{5^3}{6^3}\right)^n \\ &= \frac{1}{6}\left(\frac{5}{6}\right)^2 \frac{1}{1 - 5^3/6^3} \\ &= \frac{25}{91}.\end{aligned}$$

4. Find the exact value of

$$\lim_{x\to 3} \frac{x}{x-3} \int_3^x \sin t \; dt.$$

[NES-MAA 2007 #1] If we substitute $x = 3$ into the various pieces of the given function, we get the indeterminate $\frac{0}{0}$, which gets us nowhere. Instead, let $F(x) = \int_x^3 \sin t \; dt$. Then $F(3) = 0$ and

$$\begin{aligned} \lim_{x\to 3} \frac{x}{x-3} \int_3^x \sin t \; dt &= \lim_{x\to 3} x \cdot \lim_{x\to 3} \frac{F(x)}{x-3} \\ &= 3 \lim_{x\to 3} \frac{F(x)-F(3)}{x-3} \\ &= 3F'(3) \\ &= 3\sin 3. \end{aligned}$$

Wiles Level

5. An object moves 8 cm in a straight line from A to B, turns at an angle α, measured in radians and chosen at random from the interval $(0, \pi)$, and moves 5 cm in a straight line to C. What is the probability that $AC < 7$?

[NES-MAA 2007 #5] First we find the value of α which gives $AC = 7$. The Law of Cosines applied to $\triangle ABC$ implies that

$$7^2 = 5^2 + 8^2 - 2 \cdot 5 \cdot 8 \cos\alpha, \text{ so } \cos\alpha = \frac{1}{80}(25 + 64 - 49) = 1/2.$$

and $\alpha = \pi/3$. Thus for $\alpha < \pi/3$ we have $AC < 7$ and when $\alpha \geq \pi/3$ we have $AC \geq 7$. So the probability that $AC < 7$ is $(\pi/3)/\pi = 1/3$.

6. Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

[Putnam 2000 A2] First solution: Let a be an even integer such that a^2+1 is not prime. (For example, choose $a \equiv 2 \pmod 5$, so that $a^2 + 1$ is divisible by 5.) Then we can write $a^2 + 1$ as a difference of squares $x^2 - b^2$, by factoring $a^2 + 1$ as rs with $r \geq s > 1$, and setting $x = (r+s)/2$, $b = (r-s)/2$. Finally, put $n = x^2 - 1$, so that $n = a^2 + b^2$, $n + 1 = x^2$, $n + 2 = x^2 + 1$.

Second solution: It is well-known that the equation $x^2 - 2y^2 = 1$ has infinitely many solutions (the so-called "Pell" equation). Thus setting $n = 2y^2$ (so that $n = y^2 + y^2$, $n+1 = x^2 + 0^2$, $n+2 = x^2 + 1^2$) yields infinitely many n with the desired property.

Third solution: As in the first solution, it suffices to exhibit x such that $x^2 - 1$ is the sum of two squares. We will take $x = 3^{2^n}$, and show that $x^2 - 1$ is the sum of two squares by induction on n: if $3^{2^n} - 1 = a^2 + b^2$, then

$$\begin{aligned} (3^{2^{n+1}} - 1) &= (3^{2^n} - 1)(3^{2^n} + 1) \\ &= (3^{2^{n-1}} a + b)^2 + (a - 3^{2^{n-1}} b)^2. \end{aligned}$$