Math 280 Problems for October 16

Pythagoras Level

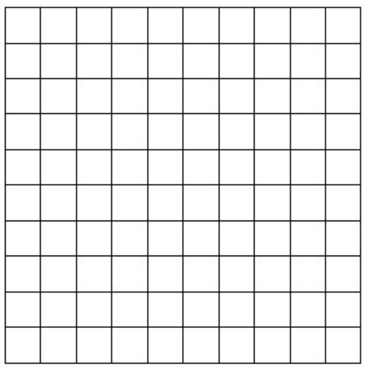
1. Show that for every sequence $x_1, ..., x_n \in (0,1)$ *at least one of the inequalities holds:*

$$x_1 \cdots x_n \leq 2^{-n}$$

or

$$(1-x_1)\cdots(1-x_n) \leq 2^{-n}.$$

2. How many squares (of all possible sizes) are there in the following picture?



Newton Level

3. Compute

$$L = \lim_{n\to\infty} \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right).$$

4. Let f be a continuous function on $[0,1]$, such that for every $x \in [0,1]$, $\int_x^1 f(t)\, dt \geq \frac{1-x^2}{2}$. Show that

$$\int_0^1 (f(x))^2\, dx \geq \frac{1}{3}.$$

Wiles Level

5. Compute

$$L = \lim_{n\to\infty} \frac{1}{n^4} \prod_{i=1}^{2n} (n^2 + i^2)^{1/n}.$$

6. Five points in the plane belong to a closed square with side 1. Prove that the distance between some two of them is at most $\sqrt{2}/2$.