Math 280 Solutions for October 16

Pythagoras Level

1. (Kansas MAA 2007 #1) Suppose the claim is not true. Then

$$x_1 \cdots x_n (1-x_1)\cdots(1-x_n) > 4^{-n}$$

But $x_i(1-x_i) \le 1/4$ for each i, so we get

$$4^{-n} < x_1 \cdots x_n (1-x_1)\cdots(1-x_n) \le 4^{-n},$$

a contradiction.

2. (Kansas MAA 2007 #5) Count the number of possible lower left end corners of squares with sidelength k, $1 \le k \le 10$. We have clearly the lower left square with sidelength $(10-k)$, this yields $(10-k+1)^2$ points. Thus the number of squares is

$$\sum_{k=1}^{10} (10-k+1)^2 = 385.$$

Newton Level

3. (Kansas MAA 2006 #1) Compute

$$S_n = \prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right) = \prod_{i=2}^{n}\frac{(i-1)(i+1)}{i^2} = \frac{n+1}{2n}.$$

Thus $\lim_{n\to\infty} S_n = 1/2$.

4. (Kansas MAA 2005 #1)

$$0 \le \int_0^1 (f(x)-x)^2\,dx = \int_0^1 (f(x))^2\,dx - 2\int_0^1 xf(x)\,dx + \int_0^1 x^2\,dx$$

It follows that

$$\int_0^1 (f(x))^2\,dx \ge 2\int_0^1 xf(x)\,dx - \frac{1}{3}.$$

But

$$\frac{1}{3} = \int_0^1 \frac{1-x^2}{2}\,dx \le \int_0^1\int_x^1 f(t)\,dt\,dx = \int_0^1\int_0^t f(t)\,dx\,dt = \int_0^1 tf(t)\,dt$$

Thus

$$\int_0^1 (f(x))^2\,dx \ge 2\int_0^1 xf(x)\,dx - \frac{1}{3} \ge \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

Wiles Level

5. (Kansas MAA 2007 #2) Let $A_n = n^{-4}\prod_{i=1}^{2n}(n^2+i^2)^{1/n}$. Then

$$\begin{aligned}\ln(A_n) &= \left(\sum_{i=1}^{2n}\frac{1}{n}\ln(n^2+i^2)\right) - 4\ln(n)\\ &= \left(\sum_{i=1}^{2n}\frac{1}{n}\ln(n^2(1+i^2/n^2))\right) - 4\ln(n)\\ &= \left(\sum_{i=1}^{2n}\frac{1}{n}(2\ln(n)+\ln((1+i^2/n^2)))\right) - 4\ln(n)\\ &= \left(\sum_{i=1}^{2n}\frac{1}{n}\ln(1+i^2/n^2)\right)\end{aligned}$$

This last expression is the Riemann sum for $\int_0^2 \ln(1+x^2)\ dx$. Thus

$$\begin{aligned}L = \lim_{n\to\infty} A_n &= \exp\left(\int_0^2 \ln(1+x^2)\ dx\right)\\ &= \exp\left(x\ln(1+x^2)|_0^2 - 2\int_0^2 \frac{x^2}{1+x^2}\ dx\right)\\ &= \exp(2\ln(5) - 2 + 2\tan^{-1}(2))\\ &= 25e^{2\tan^{-1}(2)-2}\end{aligned}$$

6. (Kansas MAA 2006 #5) Divide the unit square in four identical subsquares with sidelength 1/2 passing through the center. By the pigeonhole principle, there will be two points in one of the squares (including its sides). Therefore, the distance between them will not be more than the diameter of each of the subsquares which is $\sqrt{2}/2$.