Math 280 Problems for October 23

Pythagoras Level

1. Let $a_1 = 3$ and for $n \geq 1$, $a_{n+1} = a_n^2 - 2$. Prove that if $m \neq n$ then a_m and a_n are relatively prime.

   (NCS-MAA 1998 #4) Note first that all an are odd (by an easy induction). Assume that $m < n$. Then

   $$\begin{aligned} a_{m+1} &= a_m^2 - 2 \equiv -2 \mod a_m \\ a_{m+2} &\equiv (-2)^2 - 2 \equiv 2 \mod a_m \end{aligned}$$

   And by induction , for every $k \geq 2$, $a_{m+k} \equiv 2 \mod a_m$. Thus $a_n = qa_m + 2$ or $a_n = qa_m - 2$ for some integer q, and therefore every common factor of a_m and a_n is a divisor of 2. Since both a_m and a_n are odd, they are relatively prime.

2. Let $f_1(x) = f(x) = \frac{1}{1-x}$, and for $n > 1$, $f_n(x) = f(f_{n-1}(x))$. Evaluate $f_{2011}(2010)$.

   (NCS-MAA 1999 #2) We have

   $$f_2(x) = \frac{1}{1 - \frac{1}{1-x}} = \frac{x-1}{x},$$

   and

   $$f_3(x) = \frac{1}{1 - \frac{x-1}{x}} = x.$$

   Then $f_4(x) = f_1(x)$, and for each n, $f_n(x) = f_{n+3}(x)$. Since $2011 \equiv 1 \mod 3$, $f_{2011}(x) = f_2(x)$. So $f_{2011}(2010) = \frac{2009}{2010}$.

   Note: Composition of functions that are the quotient of two linear functions is equivalent to 2×2 matrix multiplication. In this case, the fact that $f_3(x) = x$ is related to

   $$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Newton Level

3. Find the maximum and minimum values of

   $$2x|x| - 5x + 1,$$

   for $|x + 1| \leq 3$. Justify your answer.

   (NCS-MAA 1997 #4) The maximum value is $33/8 = 4.125$, at $x = -5/4$, and the minimum value is -11, at $x = -4$. To see this, let

   $$f(x) = 2x|x| - 5x + 1 = \begin{cases} 2x^2 - 5x + 1, & x \geq 0 \\ -2x^2 - 5x + 1, & x < 0 \end{cases}$$

   Then

   $$f'(x) = \begin{cases} 4x - 5, & x \geq 0 \\ -4x - 5, & x < 0 \end{cases}$$

   The range $|x + 1| \leq 3$ is equivalent to $-3 \leq x + 1 \leq 3$; i.e., $-4 \leq x \leq 2$. Candidates for local extrema are at $x = -4$, $-5/4$, $5/4$ and 2, where $f(x)$ has values -11, $33/8 = 4.125$, $-17/8 = -2.125$ and -1, respectively. Thus the maximum value is $33/8$, at $x = -5/4$, and the minimum is -11, at $x = -4$.

4. Evaluate

$$\int_1^2 \frac{1}{\lfloor x^2 \rfloor} dx,$$

where as usual $\lfloor u \rfloor$ denotes the greatest integer less than or equal to u.

(NCS-MAA 1999 #4) From the definition of the floor function,

$$\frac{1}{\lfloor x^2 \rfloor} = \begin{cases} \frac{1}{1} & 1 \le x \le \sqrt{2} \\ \frac{1}{2} & \sqrt{2} \le x \le \sqrt{3} \\ \frac{1}{3} & \sqrt{3} \le x \le 2 \end{cases}$$

Then

$$\int_1^2 \frac{1}{\lfloor x^2 \rfloor} dx = 1(\sqrt{2} - 1) + \frac{1}{2}(\sqrt{3} - \sqrt{2}) + \frac{1}{3}(2 - \sqrt{3}) = -\frac{1}{3} + \frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3}.$$

Wiles Level

5. If

$$x = \frac{1 + \sqrt{2010}}{2},$$

what is the value of

$$(4x^3 - 2013x - 2010)^{2015}?$$

Justify your answer.

(NCS-MAA 1997 #5) We know $4x^2 - 4x + 1 = (2x - 1)^2 = 2010$, so $4x^2 = 4x + 2009$ and $4x^3 = 4x^2 + 2009x$. Then

$$\begin{aligned} 4x^3 - 2013x - 2010 &= 4x^2 + 2009x - 2013x - 2010 \\ &= 4x^2 - 4x - 2010 \\ &= 4x^2 - 4x + 1 - 2011 \\ &= 2010 - 2011 \\ &= -1. \end{aligned}$$

Thus $(4x^3 - 2013x - 2010)^{2015} = (-1)^{2015} = -1$.

6. Given that a, b and c are real numbers with $a < b$ and $a < c$, prove that

$$a < \frac{bc - a^2}{b + c - 2a} < \min\{b, c\}.$$

(NCS-MAA 1999 #6) Since $a < c$ and $b - a > 0$ then

$$\begin{aligned} a(b - a) &< c(b - a) \\ ab - a^2 &< cb - ca \\ ab + ac - a^2 &< bc \\ ab + ac - 2a^2 &< bc - a^2 \\ a &< \frac{bc - a^2}{b + c - 2a} && \text{since } b + c - 2a > 0 \end{aligned}$$

So the first inequality is true.

Note that because of the symmetry in b and c in the problem, we may assume without loss of generality that $b \le c$. Then since $a \ne b$

$$\begin{aligned} 0 &< (a^2 - 2ab + b^2) = (a - b)^2 \\ -a^2 &< -2ab + b^2 \\ bc - a^2 &< bc - 2ab + b^2 \\ \frac{bc - a^2}{b + c - 2a} &< b = \min\{b, c\} \end{aligned}$$

So the second inequality is true.