MATH 210: Prove it!

NAME:

Def 1. *A number r is rational if and only if there exists integers a and b such that $r = \frac{a}{b}$.*

Def 2. $0! = 1$ *and for all integers $n \geq 1$,*

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1.$$

Def 3. *The real number e is given by*

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Axiom 1. *There is no integer n such that $0 < n < 1$.*

Axiom 2. *The addition or multiplication of two or more integers results in an integer.*

Axiom 3. *If $0 < n < m$, then $\frac{1}{m} < \frac{1}{n}$.*

Axiom 4. $\sum_{i=0}^{\infty} a_i - \sum_{i=0}^{k} a_i = \sum_{i=k+1}^{\infty} a_i$.

Axiom 5. *If $a_i > 0$ for all i, then $\sum_{i=k}^{\infty} a_i > 0$ for all k.*

Using some or all of the above definitions and axioms (as well as simple rules of algebra and fractions), prove the following theorem:

Theorem 1. *If $r = \frac{a}{b}$ where a and $b \in \mathbb{Z}$, then*

$$b!\left(r - \sum_{n=0}^{b} \frac{1}{n!}\right) \in \mathbb{Z}.$$

Theorem 2. *For any integer $b > 0$,*

$$b!\left(e - \sum_{n=0}^{b} \frac{1}{n!}\right) > 0.$$

Theorem 3. *For integers $0 < b \leq n$,*

$$\frac{b!}{n!} \leq \frac{1}{(b+1)^{n-b}}.$$

Theorem 4. *For $b \geq 1$,*

$$\sum_{k=b+1}^{\infty} \frac{1}{(b+1)^{k-b}} = \frac{1}{b}.$$

(Hint: Let $S = \sum_{k=b+1}^{\infty} \frac{1}{(b+1)^{k-b}}$ and evaluate $(b+1)S - S$.)

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Using the theorems that your group members and you just proved, prove the following theorem.

Theorem 5. *The real number e is irrational.*