MATH 210: Construction of the Reals Homework

1. Let C^∞ be the set of all continuous functions on the interval $[0,1]$. Show that $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$ is a metric on C^∞.

2. Give two other elements (as Cauchy sequences) of the equivalence class $\{1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000} \cdots\}/ =_C$.

3. Give the element of $\mathbb{R}$ (in standard notation) that corresponds to

$$\left\{1, 1+\frac{1}{1}, 1+\frac{1}{1+\frac{1}{1}}, 1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}, \dots\right\} / =_C$$

4. A sequence $\{x_n\}$ is said to be **eventually bounded by** M iff there exists N such that for every $n > N$ we have $|x_n| < M$. Prove that if $\{x_n\}$ is a Cauchy sequence then it is eventually bounded.
(Hint: Since the sequence is Cauchy, it has a limit, call it L. Then use the limit definition of $x_n \to L$ to show that there exists N such that for $n > N$ we have $L - 1 < x_n < L + 1$. Now find the appropriate bound on $|x_n|$.)

5. Suppose $\{r_n\} =_C \{s_n\}$ and $\{t_n\} =_C \{u_n\}$. Show that
   (a) $\{r_n + t_n\} =_C \{s_n + u_n\}$
   (b) $\{r_n t_n\} =_C \{s_n u_n\}$ (Hint: use $0 = -s_n t_n + s_n t_n$ and Problem 4.)

6. Consider the sequence of rationals $\{1, 4, 16, \dots, 4^n, \dots\}$ and the metric on $\mathbb{Q}$ given by

$$d_2(x,y) = 2^{(\text{\# of 2's in the denominator of } x-y) - (\text{\# of 2's in the numerator of } x-y)}.$$

   (a) Prove that this sequence is Cauchy under the metric $d_2(x,y)$.
   (b) Show that this sequence has limit $L = 0$ under the metric $d_2(x,y)$.