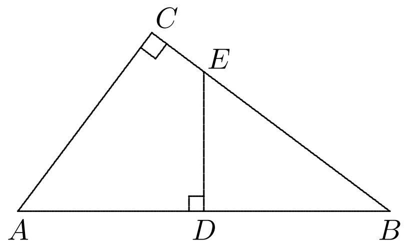
1. Quadrilateral area.

In the figure at the right $AB = 20$, $AC = 12$, $AD = DB$, angles ACB and ADE are right angles. Find the area of the quadrilateral $ADEC$.



2. Sequence sum.

A sequence begins with a_1, a_2, and for $n > 2$ is defined by $a_n = a_{n-1} - a_{n-2}$. Find the sum of the first 2004 terms (in terms of a_1 and a_2), and defend your answer.

3. Sum of cubes of roots.

If r and s are the roots of the quadratic equation

$$x^2 + ax + \frac{a^2 - 1}{2} = 0,$$

find $r^3 + s^3$ in terms of a, and express it as a polynomial in a with rational coefficients.

4. Integer linear combination.

Do there exist integers m and n satisfying

$$130m + 559n = 52?$$

If so, find such a pair (m, n). If not, explain.

5. A polynomial in x^3.

Let $P(x) = x^3 - x^2 + x - 2$. Does there exist a nontrivial polynomial $Q(x)$ with real coefficients such that the degree of every term of the product $P(x)Q(x)$ is a multiple of 3? If so, find one. If not, show there is none.

6. Shuffling cards.

A card shuffling machine always rearranges cards in the same way relative to the order in which they are given to it. The thirteen spades arranged in the order

$$A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$$

are put into the machine, shuffled, and then the shuffled cards are put into the machine and shuffled again. If at this point the order of the cards is

$$3, K, 10, 2, Q, 9, 4, J, 8, 6, 7, A, 5,$$

what was the order of the cards after the first shuffle?

7. Slanted asymptote.

Let $f(x) = 2x + \sqrt{x^2 + 4x + 5}$ for all real x. Show that as $x \to -\infty$ the graph of f is asymptotic to a nonhorizontal straight line, and find the equation of this line. (You must show rigorously that the distance between this line and the graph of f approaches zero.)

8. Find the n-th term.

The sequence $\{a_n\}$ is defined recursively by $a_0 = 2, a_1 = 671$, and for $n \geq 0$, $a_{n+2} = 671a_{n+1} - 2004a_n$. Find, and prove, a closed form expression for a_n.

9. Same fractional parts.

Let n be an integer, $n \geq 3$, and let x be a real number such that the numbers x, x^2 and x^n have the same fractional parts. Prove that x is an integer. (The fractional part of a number u is $u - \lfloor u \rfloor$; i.e., u minus the greatest integer in u.)

10. Limit of product of cosines.

The sequence of functions $\{u_n(x)\}$ is defined for real x by $u_1(x) = \cos(x/2)$ and for $n > 1$, $u_n(x) = u_{n-1}(x)\cos(x/2^n)$. Thus

$$u_n(x) = \cos\frac{x}{2}\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n}.$$

If $x = 0$, it is clear that $u_n(x) = 1$ for every n. Find $\lim_{n\to\infty} u_n(x)$ as a function of x for $x \neq 0$.