Math 280 Problems for September 3

Pythagoras Level

#1. In popular culture, many of us are familiar with the stereotype of the mad scientist. In this case, a mad veterinarian invents an animal transmogrifying machine. The machine can transmogrify:

- Two cats into one cat, or vice-versa
- One cat and one dog into one dog, or vice-versa
- Two dogs into one cat, or vice-versa

Beginning with three cats and one dog, is it possible to end up with

(a) one dog and no cats?

(b) one cat and no dogs?

Be sure to justify your answers.

[Iowa MAA 2010 #1] The given transmogrifications always either leave the number of dogs constant or change it by 2. Thus, beginning with an odd number of dogs, the mad veterinarian will always end with an odd number of dogs. It is therefore impossible to end up with one cat and no dogs. To end up with one dog and no cats, the veterinarian can use the second machine three times, among other possibilities.

#2. If the pth term of an arithmetic progression is q and the qth term is p, where $p \neq q$, find the $(p+q)$th term.

[NCS MAA 2002 #2] Let the first term be a and the common difference d. Then

$$a + (p-1)d = q$$

and

$$a + (q-1)d = p.$$

Subtract the second equation from the first to get

$$(p-q)d = q - p.$$

We are given that $p - q \neq 0$, so $d = -1$, and the $(p+q)$th term is

$$\begin{aligned} a + (p+q-1)d &= [a + (p-1)d] + qd \\ &= q + q(-1) \\ &= 0 \end{aligned}$$

Newton Level

#3. Evaluate the integral

$$I = \int_{1/2}^{2} \frac{\ln x}{1+x^2}\, dx.$$

[Missouri CMC 2010 #4]

$$\begin{aligned} I = \int_{1/2}^{1} \frac{\ln x}{1+x^2}\, dx + \int_{1}^{2} \frac{\ln x}{1+x^2}\, dx &= \int_{1/2}^{1} \frac{\ln x}{1+x^2}\, dx + \int_{1}^{1/2} \frac{-\ln u}{1+u^{-2}} \left(\frac{-1}{u^2} du\right) \\ &= \int_{1/2}^{1} \frac{\ln x}{1+x^2}\, dx - \int_{1/2}^{1} \frac{\ln u}{1+u^2}\, dx = 0. \end{aligned}$$

The key to this problem is to discover that the integral has hidden symmetry; observe that the limits are mutual reciprocals. Let $x = u^{-1}$, $dx = -u^{-2} du$. Then

$$I = \int_{2}^{1/2} \frac{-\ln u}{1+u^{-2}} \left(\frac{-1}{u^2} du\right) = -\int_{1/2}^{2} \frac{\ln u}{1+u^2} du = -I.$$

#4. A smooth function $f(x)$ has $f''(x) > 0$ for all x in $[0,1]$. For each point a in $[0,1]$, draw the tangent line to $y = f(x)$ at the point where $x = a$. Let $A(a)$ be the area bounded by the curve $y = f(x)$, the tangent line at a, $x = 0$, and $x = 1$. For what value of a is the area minimized?

[Iowa MAA 2010 #7] An equation for the tangent line is $y = f'(a) \cdot (x - a) + f(a)$. Thus we can compute (with $F(x)$ being the antiderivative of f that has $F(0) = 0$):

$$\begin{aligned} A(a) &= \int_0^1 f(x) - f'(a)(x-a) - f(a)\, dx \\ &= F(1) - f'(a)\left(\frac{(1-a)^2}{2} - \frac{a^2}{2}\right) - f(a) \\ &= F(1) - \frac{f'(a)}{2} \cdot (1-2a) - f(a) \end{aligned}$$

Thus

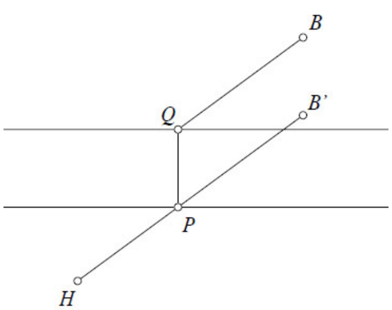
$$A'(a) = 0 - \frac{f''(a)}{2}(1-2a) + \frac{f'(a)}{2} \cdot 2 - f'(a).$$

Setting $A'(a) = 0$, we see that $a = 1/2$ is the solution, no matter what function f is! Checking, we can see that since $f'' > 0$, $A'(a)$ is negative when $a < 1/2$ and positive when $a > 1/2$, so indeed $a = 1/2$ is the absolute minimum on this interval.

Wiles Level

#5. A farmer lives in a farmhouse H on one side of a stream bounded by two parallel lines. He often has to walk to his barn B on the other side of the stream. Since he is tired of getting wet, he wants to build a bridge PQ perpendicular to the stream, with P on the same side of the stream as H. He also wants the total walking distance HP+PQ+QB to be as short as possible. How should he determine where to place the bridge?

[Iowa MAA 2010 #2] The farmer should place the bridge such that HP and QB are parallel. To do so, let w be the width of the stream. Translate B toward the stream to a point B0 a distance w away from B (with BB0 perpendicular to the stream). Then draw HB0, and where it intersects the near side of the stream, place point P, as shown in the figure.



This distance is minimal since HP+PQ+QB = HP+PQ+PB0. PQ = w is constant, and HP+PB0 is clearly minimized when P is on the straight line HB0.

#6. How many rearrangements of the string of letters *aabcde* have exactly two letters in their original places? The two *a*s are indistinguishable, so an *a* in either the first or second position is considered to be in its original place.

[Iowa MAA 2010 #8] First choose two positions to be correct.

If the two chosen positions are among the *bcde* spots (that is, positions 3 through 6), then the two *a*s must be in the remaining two of those positions (and there is only one way to do that), and the remaining two letters must occupy the positions originally held by the *a*s, and there are two ways to do that. Thus we have $\binom{4}{2} \cdot 1 \cdot 2 = 12$ ways so far.

Otherwise, there are $\binom{6}{2} - \binom{4}{2} = 9$ ways to choose two spots, at least one of which is an a. In that case, the remaining 4 spots all contain different letters, and (either by listing the 24 possibilities and counting, or by using the usual derangement arguments) there are $D_4 = 9$ ways to arrange the four letters so that none are in their original spots. Thus there are $9 \cdot 9 = 81$ ways to arrange the letters in this case.

In total we thus have $12 + 81 = 93$ ways.