Math 280 Solutions for September 10

Pythagoras Level

#1. You have an integer with many digits. This integer is one more than a multiple of 9. One of the digits is removed and the remaining digits are rearranged. The new number is 8 more than a multiple of 9. What digit was removed? Justify your answer.

[Eastern Illinois University Challenge of the Week 8/28/09 #1] Suppose M is a positive integer whose decimal expansion is

$$a_n 10^n + \cdots + a_2 10^2 + a_1 10 + a_0$$

where a_0, a_1, a_2, $\ldots$ an are integers between 0 and 9, inclusive. M being k more than a multiple of 9 is true if and only if the sum of the digits of M (i.e. $a_n + \cdots + a_2 + a_1 + a_0$) is k more than a multiple of 9. Therefore, the given problem is equivalent to:

> You have an integer. The sum of the digits of this number is one more than a multiple of 9. One of the digits is removed and now the sum is 8 more than a multiple of 9. What digit was removed?

From this equivalent formulation it is clear that the digit that was removed was 2.

#2. How many ordered triples of integers (x, y, z) satisfy the equation $|x| + |y| + |z| = 2010$?

[2010 Kansas State Math Competition #2] Let $h(n, d)$ be the number of **positive** integer solutions of $x_1 + \cdots + x_d = n$. This is the same as the number of non-negative integer solutions of $x_1 + \cdots + x_d = n - d$. This is the number of ways of arranging $n - d$ stars and $d - 1$ bars and taking x_i to be the number of stars in the i^{th} block: e.g., $**||****|*|$ corresponds to $2 + 0 + 4 + 1 + 0 = 7$. Therefore

$$h(n, d) = \binom{n-1}{d-1}.$$

If we count the solutions to $|x| + |y| + |z| = 2010$ by the number of 0s and negatives they contain, we get

$$\begin{aligned} 6h(n, 1) + 12h(n, 2) + 8h(n, 3) &= 6\binom{n}{0} + 12\binom{n-1}{1} + 8\binom{n-1}{2} \\ &= 6 + 12(n-1) + 8(n-1)(n-2)/2 \\ &= 4n^2 + 2 \end{aligned}$$

Thus the answer is $4(2010^2) + 2 = 16160402$

Newton Level

#3. Show that for every $0 < \theta \le \pi$,

$$\int_0^\theta \sqrt{1 + \cos^2(t)}\, dt > \sqrt{\theta^2 + \sin^2(\theta)}.$$

[2010 Kansas State Math Competition #1] The problem is very easy, if one realizes that the left hand side is the arc length of the curve $y = sin(x)$ from $(0, 0)$ to $(\theta, \sin(\theta))$, while the right hand side is the Euclidian distance between the same points. For an analytical solution consider the function

$$F(\theta) = \int_0^\theta \sqrt{1 + \cos^2(t)}\, dt - \sqrt{\theta^2 + \sin^2(\theta)}$$

Since $F(0) = 0$, it suffices to show $F'(\theta) > 0$. We have (by the fundamental theorem of calculus):

$$F'(\theta) = \sqrt{1 + \cos^2(\theta)} - \frac{\theta + \sin\theta\cos\theta}{\sqrt{\theta^2 + \sin^2(\theta)}}.$$

Thus, it amounts to showing

$$(1+\cos^2(\theta))(\theta^2+\sin^2(\theta)) > (\theta+\sin\theta\cos\theta)^2.$$

The last one is equivalent (after squaring) to

$$\theta^2\cos^2(\theta)+\sin^2(\theta) > 2\theta\sin\theta\cos\theta,$$

which is equivalent to $(\theta\cos(\theta)-\sin\theta)^2 > 0$, which is satisfied for all $\theta \in (0,\pi]$.

#4. Find the area of the set of all points in the unit square, which are closer to the center of the square than to its sides.

[2010 Kansas State Math Competition #4] Let us center the coordinate system at the center of the square, with axes parallel to the sides. The set obviously has a lot of symmetries. Consider the portion in the set $y \geq |x|$, which is obviously a quarter of the whole set. The set may be described analytically by the equations

$$\sqrt{x^2+y^2} \leq \frac{1}{2}-y \qquad \text{and} \qquad y \geq |x|$$

which leads us to

$$|x| \leq y \leq \frac{1}{4}-x^2$$

Thus,

$$S = 4\int_{-\frac{\sqrt{2}-1}{2}}^{\frac{\sqrt{2}-1}{2}} \frac{1}{4}-x^2-|x|\,dx = 8\int_0^{\frac{\sqrt{2}-1}{2}} \frac{1}{4}-x^2-x\,dx = \frac{4\sqrt{2}-5}{3}.$$

Wiles Level

#5. Let f_n denote the nth Fibonacci number: $f_0=0$, $f_1=1$, and $f_n=f_{n-1}+f_{n-2}$ for $n\geq 2$. Fix $c \geq 2$. Evaluate

$$\sum_{n=1}^{\infty}\frac{f_n}{c^n}.$$

[2010 Kansas State Math Competition #3] Call this number A. Then

$$\begin{aligned}
A &= \sum_{n=1}^{\infty}\frac{f_n}{c^n} \\
&= \frac{1}{c}+\sum_{n=2}^{\infty}\frac{f_{n-1}+f_{n-2}}{c^n} \\
&= \frac{1}{c}+\sum_{n=2}^{\infty}\frac{f_{n-1}}{c^n}+\sum_{n=2}^{\infty}\frac{f_{n-2}}{c^n} \\
&= \frac{1}{c}+\frac{1}{c}\sum_{n=2}^{\infty}\frac{f_{n-1}}{c^{n-1}}+\frac{1}{c^2}\sum_{n=2}^{\infty}\frac{f_{n-2}}{c^{n-2}} \\
&= \frac{1}{c}+\frac{1}{c}\sum_{n=1}^{\infty}\frac{f_n}{c^n}+\frac{1}{c^2}\sum_{n=1}^{\infty}\frac{f_n}{c^n} \\
&= \frac{1}{c}+\frac{A}{c}+\frac{A}{c^2}
\end{aligned}$$

Solving yields

$$A = \frac{c}{c^2-c-1}.$$

#6. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

[1998 Putnam A-1] Consider the plane containing both the axis of the cone and two opposite vertices of the cube's bottom face. The cross section of the cone and the cube in this plane consists of a rectangle of sides s and $s\sqrt{2}$ inscribed in an isosceles triangle of base 2 and height 3, where s is the side-length of the cube. (The $s\sqrt{2}$ side of the rectangle lies on the base of the triangle.) Similar triangles yield $s/3 = (1-s\sqrt{2}/2)/1$, or $s = (9\sqrt{2}-6)/7$.