Math 280 Solutions for September 24

Pythagoras Level

#1. Find a positive integer the first digit of which is 1, which has the property that if this digit is moved to the end of the number, the number is tripled.

[2005 Ohio MAA CONSTUM #1] Suppose the integer is

$$n = 1 \cdot 10^m + d_{m-1} \cdot 10^{m-1} + \cdots + d_1 10 + d_0$$

Then

$$d_{m-1} \cdot 10^m + \cdots + d_1 \cdot 10^2 + d_0 \cdot 10 + 1 = 3(1 \cdot 10^m + d_{m-1} \cdot 10^{m-1} + \cdots + d_1 10 + d_0)$$

Hence

$$d_{m-1}(10^m - 3 \cdot 10^{m-1}) + \cdots + d_1(10^2 - 3 \cdot 10) + d_0(10 - 3) = 3 \cdot 10^{m-1}$$

or

$$7d_{m-1} \cdot 10^{m-1} + \cdots + 7d_1 \cdot 10 + 7d_0 = 3 \cdot 10^{m-1}$$
$$d_{m-1} \cdot 10^{m-1} + \cdots + d_1 \cdot 10 + d_0 = \frac{3 \cdot 10^{m-1}}{7}$$

Thus

$$n - 10^m = \frac{3 \cdot 10^{m-1}}{7}$$

which implies

$$n = \frac{10^{m+1} - 1}{7} = \frac{99999\ldots}{7}$$

The fewest number of 9s you need for n to be an integer is 6, and in this case we have $n = 142857$.

#2. Let $n \geq 1$ and define $A = \{1, 2, \ldots, n\}$. Denote the power set of A (i.e. the set of all subsets of A) by $P(A)$. For each subset $K \subseteq A$, define the following function:
$a(K) =$ the alternating sum of the members of K, starting with the largest element and continuing in decreasing order. For example, $a(\{1, 4, 6, 7, 9\}) = 9 - 7 + 6 - 4 + 1$ Find the following sum (justify your answer)

$$\sum_{K \in P(A)} a(K)$$

[2005 Ohio MAA CONSTUM #10] Let $P(A) = B \cup C$ where $B =$ the subsets containing n and $C =$ the subsets not containing n. Then $|B| = |C| = 2^{n1}$ and there is a bijection between the elements of B and C via

$$\{a_1, a_2, \ldots, a_j\} \leftrightarrow \{a_1, a_2, \ldots, a_j, n\}$$

The combined alternating sum of the two sets above is n and thus

$$\sum_{K \in P(A)} a(K) = n 2^{n-1}.$$

#3. The graph of a non-negative, differentiable function f divides the triangle with vertices $(0, 0)$, $(x, 0)$, and $(x, f(x))$ into two parts having equal areas for each positive value of x. Find an explicit expression for $f(x)$ if $f(2010) = 2010$.

[ISMAA 1998 #6] The area under $y = f(x)$ is half of the area of the triangle. Therefore,

$$\int_0^x f(x)\, dx = \frac{1}{2}\left(\frac{1}{2} x f(x)\right).$$

Differentiating this expression by using the Fundamental Theorem of Calculus yields,

$$f(x) = \frac{1}{4}(f(x) + x f'(x)).$$

This gives the differential equation $y = \frac{1}{4}(y + xy')$ or $3y = xy'$. This equation can be solved by separation of varialbles to give $y = cx^3$, for some constant c. Finally, using the initial condition gives $f(x) = x^3/(2010)^2$.

#4. Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

[Putnam 2005 B3] The functions are precisely $f(x) = cx^d$ for $c, d > 0$ arbitrary except that we must take $c = 1$ in case $d = 1$. To see this, substitute a/x for x in the given equation:

$$f'(x) = \frac{a}{xf(a/x)}.$$

Differentiate:

$$f''(x) = -\frac{a}{x^2 f(a/x)} + \frac{a^2 f'(a/x)}{x^3 f(a/x)^2}.$$

Now substitute to eliminate evaluations at a/x:

$$f''(x) = -\frac{f'(x)}{x} + \frac{f'(x)^2}{f(x)}.$$

Clear denominators:

$$xf(x)f''(x) + f(x)f'(x) = xf'(x)^2.$$

Divide through by $f(x)^2$ and rearrange:

$$0 = \frac{f'(x)}{f(x)} + \frac{xf''(x)}{f(x)} - \frac{xf'(x)^2}{f(x)^2}.$$

The right side is the derivative of $xf'(x)/f(x)$, so that quantity is constant. That is, for some d,

$$\frac{f'(x)}{f(x)} = \frac{d}{x}.$$

Integrating yields $f(x) = cx^d$, as desired.

#5. Consider the numbers

$$a_2 = 11,\ a_3 = 111,\ a_4 = 1111,\ a_5 = 11111,\ \ldots$$

Show that if n is composite, then so is a_n.

[2005 Ohio MAA CONSTUM #6] Suppose k is composite. Let $k = mn$ be a nontrival factorization. Then

$$a_k = a_m \times \sum_{i=0}^{n-1} 10^{mi},$$

so a_k is composite.

#6. Suppose that $a, b \in \mathbb{R}$ with $a < b$. Suppose that $f : (a, b) \to \mathbb{R}$. Suppose that f is increasing and satisfies the property that for all $\lambda \in (0, 1)$ and $x, y \in (a, b)$

$$f(\lambda x + (1 - \lambda)y)\lambda f(x) + (1 - \lambda)f(y)$$

Prove that f is continuous on (a, b).

[ICMC 2009 #6] The condition to which f is subject implies that for $r < s < t$,

$$\frac{f(s) - f(r)}{s - r} \le \frac{f(t) - f(r)}{t - r} \le \frac{f(t) - f(s)}{t - s}$$

Now let $\epsilon > 0$. Let $x_0 \in (s, t) \subset (a, b)$. Choose $w \in \mathbb{N}$ large enough so that $(x_0 - \epsilon/w, x_0 + \epsilon/w) \subset (s, t)$. Let $m = \frac{f(t)-f(x_0)}{t-x_0}$. Let k be equal to the larger of w or m. Finally, let $\delta = \epsilon/k$. If $|x - x_0| < \delta$, the inequality above implies that $|f(x) - f(x_0)| < \epsilon$.