Math 280 Problems for October 1

Pythagoras Level

#1. Suppose that $a_1, a_2, a_3, \ldots, a_n, \ldots,$ is an increasing sequence of positive integers such that $a_{n+1} = a_n + a_{n1}$ for $n \geq 2$ and $a_7 = 100$. Determine the value of a_8.

#2. Suppose the following system of linear equations has no solutions. Find k.

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= k \\ x + y + kz &= k^2 \end{aligned}$$

Newton Level

#3. Let a be a positive integer. In terms of a, determine the value of

$$A = \lim_{x \to 0} x + \cfrac{a}{x + \cfrac{a}{x + \cfrac{a}{x + \cfrac{a}{\cdots}}}}$$

#4. Determine the numerical value of

$$\int_0^{\pi/2} \frac{\cos(x)}{\sin(x) + \cos(x)}\, dx.$$

Wiles Level

#5. Let S be a set which is closed under the binary operation $\circ$ with the following properties:
(1) There is an element $e \in S$ such that $a \circ e = e \circ a = a$ for each $a \in S$.
(2) $(a \circ b) \circ (c \circ d) = (a \circ d) \circ (c \circ b)$ for all $a, b, c, d \in S$.
Prove or disprove the following statements:
(a) $\circ$ is associative on S.
(b) $\circ$ is commutative on S.

#6. Evaluate $\displaystyle\lim_{k \to \infty} \frac{R_k(2)}{R_k(3)}$, where

$$R_k(n) = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{n}}}}}}$$

is defined using k square-roots. Hint: Trigonometry.