Math 280 Solutions for October 22

Pythagoras Level

Problem 1: [Ohio MAA 2005 #2] Use the Binomial Theorem to solve

$$\sum_{i=0}^{2010} \binom{2010}{i} 4^{\frac{i}{2}} = \sum_{i=0}^{2010} \binom{2010}{i} 2^i = (1+2)^{2010} = 3^{10\cdot 201}$$

So $x = 3^{10}$.

Problem 2: [Ohio MAA 2005 #4] If $n \leq 2$ then automatically $f : \{1, ..., n\} \to \{1, 2, 3\}$ is not onto, so the probability is 1. Now let $n \geq 3$. Let $E1$ be the set of functions $f : \{1, ..., n\} \to \{1, 2, 3\}$ for which the element 1 is not in the range of f. Similarly we define $E2$ and $E3$. Then "Not being onto" means $f \in E1 \cup E2 \cup E3$. By inclusion-exclusion,

$$|E1 \cup E2 \cup E3| = |E1| + |E2| + |E3| - |E1 \cap E2| - |E1 \cap E3| - |E2 \cap E3| + |E1 \cap E2 \cap E3| =$$

$$2^n + 2^n + 2^n - 1 - 1 - 1 + 0 = 3(2^n - 1)$$

The probability Π of f not being surjective is, therefore,

$$\Pi = \frac{3(2^n - 1)}{3^n} = \frac{2^n - 1}{3^{n-1}}.$$

Newton Level

Problem 3: [Ohio MAA 2005 #7]

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}(e^{\frac{1}{n}} + e^{\frac{2}{n}} + e^{\frac{3}{n}} + \cdots + e^{\frac{n}{n}}) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (e^{1/n})^k \\
&= \lim_{n\to\infty} \frac{1}{n} \frac{e^{1/n} - e^{(n+1)/n}}{1 - e^{1/n}} \\
&= (1-e) \lim_{n\to\infty} \frac{\frac{e^{1/n}}{n}}{1 - e^{1/n}} \\
&= (1-e) \lim_{n\to\infty} \frac{\frac{ne^{1/n}(-1/n^2) - e^{1/n}}{n^2}}{-e^{1/n}(-1/n^2)} \\
&= (1-e) \lim_{n\to\infty} \frac{-e^{1/n}/n - e^{1/n}}{e^{1/n}} \\
&= (1-e) \lim_{n\to\infty} \left(-\frac{1}{n} - 1\right) \\
&= (e-1)
\end{aligned}$$

Problem 4: [Nick's Math Puzzles #148] We write $\sqrt{\frac{1+x}{1-x}}$ as $(1+x)(1-x^2)^{-1/2}$, and expand the latter term as a binomial series. We have

$$(1-x^2)^{-1/2} = 1 + (-1/2)(-x^2) + \frac{(-1/2)(-3/2)}{2!}(-x^2)^2 + ... + \frac{(-1/2)(-3/2)\cdots(-(2n-1)/2)}{n!}(-x^2)^n + ...$$

The coefficient of the general term, x^{2n}, is given by

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!} = \frac{(2n-1)!!}{(2n)!!}$$

Therefore, the power series expansion of

$$\sqrt{\frac{1+x}{1-x}} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!}(x^{2n} + x^{2n+1})$$

Wiles Level

Problem 5: [Ohio MAA 2005 #9] For $f(x) = 1 - x^n$, the equation of the tangent line to f at the point $(t, f(t))$ is $y - (1 - t^n) = -nt^{n-1}(x - t)$. The x and y intercepts of this line are $\frac{(n-1)t^n + 1}{nt^{n-1}}$ and $(n-1)t^n + 1$ respectively. Thus the area of the triangle is

$$A_t = \frac{1}{2} \cdot \frac{(n-1)t^n + 1}{nt^{n-1}} \cdot ((n-1)t^n + 1)$$

Differentiating A_t with respect to t gives

$$\frac{dA}{dt} = \frac{n-1}{2nt^n}\left((n^2-1)t^{2n} + 2t^n - 1\right)$$

Setting the term inside the brackets above to zero yields

$$t^n = \frac{-1 \pm n}{n^2 - 1}$$

The term

$$t^n = \frac{-1 - n}{n^2 - 1}$$

yields either negative or complex solutions, so the solution is

$$t^n = \frac{-1 + n}{n^2 - 1} = \frac{1}{n+1}$$

and thus

$$t =^n \sqrt{\frac{1}{n+1}}$$

Problem 6: [Putnam 1995 A-1] Suppose on the contrary that there exist $t_1, t_2 \in T$ with $t_1 t_2 \in U$ and $u_1, u_2 \in U$ with $u_1 u_2 \in T$. Then $(t_1 t_2) u_1 u_2 \in U$ while $t_1 t_2 (u_1 u_2) \in T$, contradiction.