Math 280 Solutions for October 29

Pythagoras Level

#1.

[Missouri MAA 2002 II-1] Let W_n denote the number of ways that n zombies can be returned to n boxes with no zombie finding its correct home. Suppose box 1 contains zombie N ($N \neq 1$). There are then two cases.

CASE 1. Box N contains zombie 1. Hence, the remaining $n-2$ zombies are to be distributed among $n-2$ boxes, with no zombie finding its correct home. There are W_{n-2} ways for this.

CASE 2. Box N doesn't contain zombie 1. Then $n-1$ zombies are to be similarly distributed among $n-1$ boxes; there are W_{n-1} ways here.

Since the choice of box 1 was arbitrary among the $n-1$ zombies 2, 3, ..., n, the total number of ways is

$$W_n = (n-1)(W_{n-2} + W_{n-1}).$$

We have, clearly, $W_1 = 0$ and $W_2 = 1$. The recursion formula then gives, successively, $W_3 = 2$, $W_4 = 9$, $W_5 = 44$, $W_6 = 265$, and finally $W_7 = 1854$. But the number of permutations of 7 zombies in 7 boxes is $7! = 5040$, so the desired probability is

$$P_7 = \frac{W_7}{7!} = \frac{1854}{5040} = \frac{103}{280} = 0.3679.$$

#2.

[Missouri MAA 2004 I-2] Note that if the two zombies picked and shot are both positive or both negative, their product and the zombie replacing the two zombies is positive. Also, if the two zombies picked and erased are of opposite sign, their product is negative. Thus, the number of negative zombies at any one time is always even. Therefore, the last zombie is positive.

Let

$$P = (1 \cdot 2 \cdot 3 \cdots 2010) \cdot (-1 \cdot -2 \cdot -3 \cdots -2010).$$

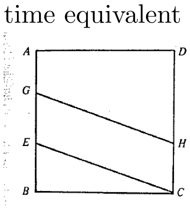
P is obviously positive. If two zombies x, y are removed from the list and the product xy is added to the list, then the product of the elements of the new list is still P. This could continue until only one element is left, and this element must necessarily be P which is positive.

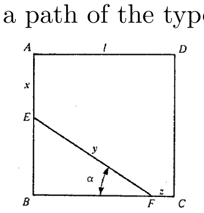
From *Quantum,* "Problems Teach Us How to Think," 11.3 (Jan/Feb 2001), p. 43 by V. Proizvolov.

Newton Level

#3.

[Missouri MAA 1996 II-4] Let the square field be denoted by $ABCD$, with the zombie initially at A and desirous of reaching C. The path of least time can evidently be described as follows. The zombie walks from A to E (a point on side AB), crawls from E to F where F is on BC, and then walks from F to C. Note that a path like $AGHC$ is time equivalent to a path of the type described with $F = C$.





Let $\overline{AE} = x$, $\overline{EF} = y$, $\overline{FC} = z$. Then the time T is given by $T = (x+z)/w + (y/s)$. If the sum $x+z$ is fixed, then the sum $y \sin\alpha + y\cos\alpha$ is also fixed, and y is minimal when $(\sin\alpha + \cos\alpha)$ is maximal. This maximum is attained for $\alpha = 45°$.

Thus for a minimal time path, $x = z$ and $y = \sqrt{2}(l-x)$, where l is the length of a side of the field. Accordingly, we have to minimize $T = (2x/w) + \sqrt{2}(l-x)/s$ for $0 \le x \le l$.

But T is a linear function of x, so its maximum occurs at an endpoint of the interval. If $x = 0$, $T = \sqrt{2}l/s$, and if $x = l$, $T = 2l/w$.

If $\sqrt{2}l/s < 2l/w$ then $w/s < \sqrt{2}$, and conversely. Hence, if $w/s < \sqrt{2}$ the minimal path is unique and the zombie should crawl diagonally across the field from A to C. If $w/s > \sqrt{2}$, the zombie should walk from A to B to C. Finally, if $w/s = \sqrt{2}$, T is independent of x and there are infinitely many minimizing paths, in fact any path $AEFC$ for which $\alpha = 45°$.

From the International Mathematical Olympiad, 1977, 2

#4. [Missouri MAA 2005 I-1] Let $(a, 7-a^2)$, $a > 0$, be a point in the first quadrant on the graph of the function $y = 7 - x^2$. The slope of the tangent line at this point is $-2a$, and the equation of the tangent line is $y - (7-a^2) = -2a(x-a)$. The x-intercept of this line is $((a^2+7)/(2a), 0)$, and the y-intercept is $(0, a^2+7)$. The distance squared between these two points is

$$s^2 = \frac{(a^2+7)^2}{4a^2} + (a^2+7)^2 = \frac{(a^2+7)^2(4a^2+1)}{4a^2}.$$

The derivative of s^2 with respect to a is

$$\frac{(a^2+7)(8a^4+a^2-7)}{2a^3},$$

and this factors as

$$\frac{(a^2+7)(8a^2-7)(a^2+1)}{2a^3}.$$

The critical point in the first quadrant occurs at

$$a = \sqrt{\frac{7}{8}} = \frac{\sqrt{14}}{4}.$$

Checking the first or second derivative will show that this critical value produces a minimum value for s^2, so the point that makes the distance between the x- and y- intercepts of the tangent line minimum is

$$\left(\frac{\sqrt{14}}{4}, \frac{49}{8}\right).$$

Wiles Level

#5.

[Missouri MAA 2004 II-1] Because f is continuous, it attains its minimum and maximum at points a and b, both in $[0,1]$, giving

$$f(a)\int_0^1 x^2 dx \le \int_0^1 x^2 f(x)\,dx \le f(b)\int_0^1 x^2\,dx$$

or

$$f(a) \le 3\int_0^1 x^2 f(x)\,dx \le f(b).$$

Thus, the Intermediate Value Theorem guarantees a point $\xi \in [0,1]$ such that

$$f(\xi) = 3\int_0^1 x^2 f(x)\,dx.$$

From *Crux Mathematicorum*, Problem 2384, November, 1998. Proposed by Paul Bracken, CRM, Université de Montréal, Québec. Solution by Michel Bataille, Rouen, France.

#6.

[Missouri 2008 II-4] Let

$$F(x) = \frac{2x+3}{x^2-2x+2}.$$

$$\frac{2x+3}{x^2-2x+2} = \sum_{k=0}^{\infty} a_k x^k$$

for all x which satisfy $|x| < \sqrt{2}$. This gives us

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} a_k(1)^k = F(1) \quad \text{and} \quad \sum_{k=0}^{\infty}(-1)^k a_k = \sum_{k=0}^{\infty} a_k(-1)^k = F(-1).$$

Thus,

$$\sum_{k=0}^{\infty} a_{2k+1} = \frac{1}{2}\Big((a_0+a_1+a_2+a_3+\cdots) - (a_0-a_1+a_2-a_3+\cdots)\Big)$$
$$= \frac{1}{2}\left(\sum_{k=0}^{\infty} a_k - \sum_{k=0}^{\infty}(-1)^k a_k\right) = \frac{1}{2}\Big(F(1) - F(-1)\Big)$$
$$= \frac{5-\frac{1}{5}}{2} = \frac{12}{5}.$$