Singular Moduli of Shimura Curves

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Based on dissertation work done while at the
University of Maryland
under the direction of
Stephen Kudla

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Classical Set-Up

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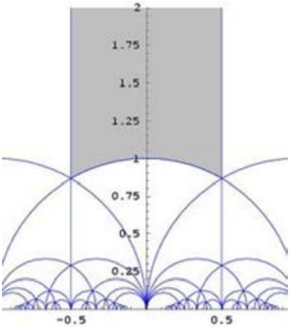

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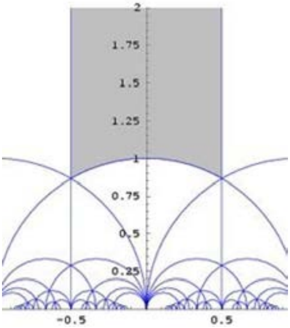

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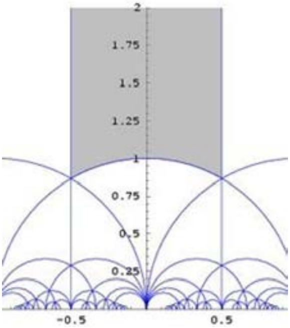

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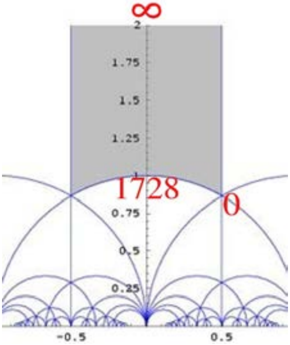

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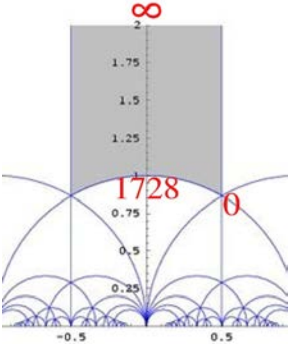
- $j(\tau) = 1/\mathbf{q} + 744 + 196884\mathbf{q} + \ldots \in \frac{1}{\mathbf{q}}\mathbb{Z}[[\mathbf{q}]]$ where $\mathbf{q} = e^{2\pi i\tau}$. (Gauss, Dedekind, Klein, etc.)



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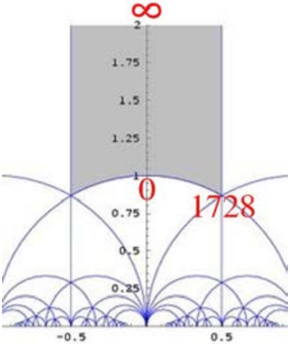

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- A CM-point τ is the solution to an integral quadratic equation with negative discriminant Δ.

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Examples

$$j\left(\frac{1+\sqrt{-3}}{2}\right)=0, \quad j(\sqrt{-5})=2^3(25+13\sqrt{5})^3$$
$$j(\sqrt{-6})=12^3(1+\sqrt{2})^2(5+2\sqrt{2})^3$$
$$j(\sqrt{-14})=2^3\left(323+228\sqrt{2}+(231+161\sqrt{2})\sqrt{2\sqrt{2}-1}\right)^3$$

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Theorem (B. Gross and D. Zagier, 1985)

$$|j(\tau_{\Delta_1}) - j(\tau_{\Delta_2})| = \prod_{n \in N(\Delta_1, \Delta_2)} n^{\varepsilon(n)}$$

where n, $\varepsilon(n) \in \mathbb{Z}$, $N(\Delta_1, \Delta_2) \subset \mathbb{Z}$ *can all be defined explicitly.*

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- Recall $j\left(\frac{1+\sqrt{-3}}{2}\right) = 0$ and $j\left(\sqrt{-1}\right) = 12^3$.

Gross-Zagier Theorem

Table 1. *Factorizations of* $N_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}(j) = \pm a^3$, $N_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}(j - 1728) = \pm b^2 d$

$\lvert d\rvert$	h	a	b
3	1	0	$2^3 3$
4	1	$2^2 3$	0
7	1	$3 \cdot 5$	3^3
8	1	$2^2 5$	$2^2 7$
11	1	2^5	$2^3 7$
19	1	$2^5 3$	$2^3 3^3$
43	1	$2^6 3 \cdot 5$	$2^3 3^4 7$
67	1	$2^5 3 \cdot 5 \cdot 11$	$2^3 3^3 7 \cdot 31$
163	1	$2^6 3 \cdot 5 \cdot 23 \cdot 29$	$2^3 3^3 7 \cdot 11 \cdot 19 \cdot 127$
23	3	$5^3 11 \cdot 17$	$7^3 11^2 19$
31	3	$3^3 11 \cdot 17 \cdot 23$	$3^{10} 11^2$
59	3	$2^{16} 11$	$2^9 11^2 23 \cdot 43$
83	3	$2^{16} 5^3$	$2^9 19 \cdot 47 \cdot 67 \cdot 79$
107	3	$2^{15} 5^3 17$	$2^9 7^3 43 \cdot 71 \cdot 103$
139	3	$2^{16} 3^3 23$	$2^9 3^{11} 103$
211	3	$2^{17} 3^3 17 \cdot 29$	$2^9 3^9 7^3 23 \cdot 67$
283	3	$2^{15} 3^3 5^3 53$	$2^9 3^{10} 19^2 31 \cdot 139$
307	3	$2^{17} 3^3 5^3 47$	$2^9 3^{11} 23 \cdot 163 \cdot 271$
331	3	$2^{15} 3^3 11 \cdot 23 \cdot 29 \cdot 59$	$2^9 3^{11} 7^3 11^2 59^2$
379	3	$2^{17} 3^3 11 \cdot 17 \cdot 53 \cdot 71$	$2^9 3^9 7^4 11^2 31 \cdot 47^2$
499	3	$2^{16} 3^3 17 \cdot 23 \cdot 41 \cdot 71 \cdot 83$	$2^9 3^{11} 7^3 71^2 463$
547	3	$2^{15} 3^3 5^3 17 \cdot 23 \cdot 101$	$2^9 3^{11} 7^3 31^2 59 \cdot 223$
643	3	$2^{15} 3^3 5^3 11 \cdot 17^3 113$	$2^9 3^{11} 11^2 43 \cdot 67 \cdot 71 \cdot 499 \cdot 607$
883	3	$2^{15} 3^3 5^3 11^2 41 \cdot 89 \cdot 113$	$2^9 3^{11} 7^3 11 \cdot 23 \cdot 43^2 307 \cdot 739$

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\begin{aligned}
|j(\sqrt{-5})| &= 2^{12} 5^3 11^3 \\
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- The factorization is a lot of small primes to large powers.

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- There is an embedding $B \hookrightarrow \mathrm{M}_2(\mathbb{Q}(\sqrt{b}))$ via

$$\alpha \mapsto \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}, \qquad \beta \mapsto \begin{pmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{pmatrix}$$

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$\mathbb{Z}[1, \alpha, \beta, \alpha\beta] \subset \mathbb{Z}\left[1, \frac{4\alpha-\alpha\beta}{5}, \frac{5-3\alpha+2\alpha\beta}{10}, \frac{4\alpha+5\beta-\alpha\beta}{10}\right]$ in B_6.

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CM Points	CM points
$j(\tau_\Delta)$ algebraic integer	$t_D(\tau_\Delta)$ algebraic
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Goal: Compute $|t_D(\tau_\Delta)|$ for $D > 1$.

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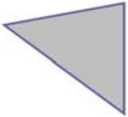

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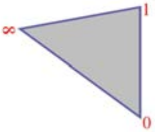

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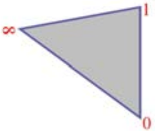
- Elkies (1998) uses geometric involutions on the covering curves $\mathcal{X}_6^*(N)$.



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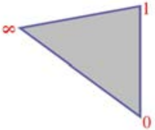

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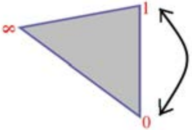

Elkies's Results

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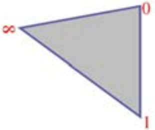

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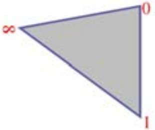

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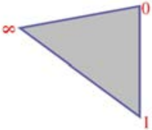

Example

$$t_6(\tau_{-163}) \stackrel{?}{=} \frac{3^{11}7^4 19^4 23^4}{2^{10}5^6 11^6 17^6}.$$

Vector-valued Modular Forms

Definition

Suppose ρ is a representation of $\tilde{\Gamma}$ on a finite dimensional complex vector space $\mathcal{V}$. Then $F : \mathfrak{h}^{\pm} \to \mathcal{V}$ is a vector-valued modular form on $\tilde{\Gamma}$ of weight k and type ρ if it satsifies, for all $\gamma \in \tilde{\Gamma}$,

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Theorem

Let L be a lattice with inner product and $\{e_\lambda\}$ a basis of $\mathbb{C}[L^\vee/L]$. Suppose f is a scalar-valued weight k modular form on $\widetilde{\Gamma_0}(N)$ with character χ_L. Then

$$F_f(\tau) = \sum_{\gamma \in \widetilde{\Gamma_0}(N) \backslash \widetilde{SL_2}(\mathbb{Z})} f|_\gamma^k(\tau) \rho_L(\gamma^{-1}) e_0.$$

is a modular form of weight k and type ρ_L, the Weil representation.

Borcherds Forms

Definition

Borcherds Form: Given a lattice L with an inner product and a modular form $F : \mathfrak{h}^{\pm} \longrightarrow \mathbb{C}[L^{\vee}/L]$, Borcherds (1998) constructed

$$\Psi(F) : \mathcal{X}_D^* \longrightarrow \mathbb{P}^1$$

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In reality:

- $\Psi(F)$ is a weight k modular form on $\mathcal{X}_D^*$ where k is determined by properties of F. So we just make sure it has weight 0.
- $\Psi(F)$ is a regularized theta lift
- $\Psi(F)$ is highly incomputable

Properties of Borcherds Forms

Theorem

If F has Fourier expansion

$$F(\tau) = \sum_{\lambda \in L^\vee / L} \sum_{m \in \mathbb{Q}} c_\lambda(m) \mathbf{q}^m e_\lambda$$

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- *The weight of* $\Psi(F)$ *is* $c_0(0)$.
- *The divisor of* $\Psi(F)$ *is given by*

$$div(\Psi(F)) = \sum_{\lambda} \sum_{m<0} c_\lambda(m) Z(-m, \lambda)$$

where the $Z(-m, \lambda)$ *are rational quadratic divisors.*

Borcherds Forms at CM Points

Theorem (J. Schofer, 2005)

$$\sum_{\substack{\text{Galois Orbit} \\ \text{of a CM Point}}} \log ||\Psi(F)|| = \frac{|Z_\Delta|}{2^{d(B)}} \sum_{\lambda \in L^\vee/L} \sum_{m<0} c_\lambda(m)\kappa_\lambda(m) \tag{1}$$

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The map t_6 is a Borcherds form. And so is t_{10}.

The map t_6 as a Borcherds Form

- Through the vectorization process, the scalar-valued $\Gamma_0(12)$ modular form

$$-6\frac{\eta_2\eta_3^2\eta_4^4\eta_6^4}{\eta_{12}^{10}} - 2\frac{\eta_2^{12}\eta_3}{\eta_1^5\eta_4^4\eta_6\eta_{12}^2} - 2\frac{\eta_2^5}{\eta_1^2\eta_4^2}$$

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- This implies there exists a nonzero constant k_6 such that

$$\Psi(F_6) = k_6 t_6.$$

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-75	11^4	$2^{10} 3^3 5$
-312	$7^4 23^4$	$5^6 11^6$
-372	$-2^2 7^4 19^4 31^2$	$3^3 5^6 11^6$
-408	$-7^4 11^4 31^4$	$3^6 5^6 17^3$
-123	$-7^4 19^4$	$2^{10} 5^6$
-147	$-11^4 23^4$	$2^{10} 3^3 5^6 7$
-163	$3^{11} 7^4 19^4 23^4$	$2^{10} 5^6 11^6 17^6$
-708	$2^8 7^4 11^4 47^4 59^2$	$5^6 17^6 29^6$
-267	$-7^4 31^4 43^4$	$2^{16} 5^6 11^6$

Coordinates of Rational CM Points on $\mathcal{X}_{10}^*$

Δ	Numerator	Denominator
-40	3^3	1
-52	$-2 \cdot 3^3$	5^2
-72	5^3	$3 \cdot 7^2$
-120	-3^3	7^2
-88	$3^3 5^3$	$2 \cdot 7^2$
-27	$-2^6 3$	5^2
-35	2^6	7
-148	$2 \cdot 3^3 11^3$	$5^2 7^2 13^2$
-43	$2^6 3^3$	$5^2 7^2$
-180	$-2 \cdot 11^3$	13^2
-232	$3^3 11^3 17^3$	$2^2 5^2 7^2 23^2$
-67	$-2^6 3^3 5^3$	$7^2 13^2$
-280	$3^3 11^3$	$2 \cdot 7 \cdot 23^2$
-340	$2 \cdot 3^3 23^3$	$7^2 29^2$
-115	$2^9 3^3$	$13^2 23$
-520	$3^3 29^3$	$2^3 7^2 13 \cdot 47^2$
-163	$-2^9 3^3 5^3 11^3$	$7^2 13^2 29^2 31^2$
-760	$3^3 17^3 47^3$	$7^2 31^2 71^2$
-235	$2^6 3^3 17^3$	$7^2 37^2 47$

Results

- The maps t_6 and t_{10} are Borcherds Forms.
- Proved all the conjectural values in Elkies's table of rational CM points of $\mathcal{X}_6^*$
- Also proved all the conjectural values in Elkies's table of rational CM points of $\mathcal{X}_{10}^*$.
- Can compute examples far beyond the scope of Elkies's work, such as norms of irrational CM points of arbitrary discriminant on $\mathcal{X}_6^*$ and $\mathcal{X}_{10}^*$.

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Example

$$|t_6(\tau_{-996})| = \frac{2^{16}7^{12}71^4 83^2}{17^6 29^6 41^6}.$$