Constructing Vector-Valued Modular Forms From Scalar Ones

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Run-of-the-Mill Modular Forms with Type

Let:

- $\mathfrak{h} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ be the upper half plane.
- $\mathrm{SL}_2(\mathbb{Z})$ act on $\mathfrak{h}$ via linear fractional transformations, i.e.
$$\gamma z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}$$
- $j(\gamma, z) = cz + d$

and choose

- A character (1-dim. rep.), $\chi : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}^\times$
- A subgroup, $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$

Example: $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\}$

Definition: A function $f : \mathfrak{h} \to \mathbb{C}$ is a modular form on Γ of weight $k \in \mathbb{Z}$ and type χ if it satifies for every $\gamma \in \Gamma$
$$f(\gamma z) = j(\gamma, z)^k \chi(\gamma) f(z).$$

Examples:
$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \qquad \text{where } q = e^{2\pi i z}$$
$$E_2(z, \chi_3) = \sum_{n \geq 1} q^n \sum_{d|n} d^2 \chi_3(n/d) \quad \text{where } \chi_3(\cdot) = \left(\frac{\cdot}{3}\right)$$

Slash Operator

Define

$$(f|_{k,\chi}\gamma)(z) = j(\gamma, z)^{-k} f(\gamma z)$$

Then

$$\begin{array}{c} f \text{ Modular Form} \\ \text{type } \chi\text{, weight } k \\ \text{on } \Gamma \end{array} \iff \begin{array}{c} f|_{k,\chi}\gamma = \chi(\gamma) f \\ \text{for all } \gamma \in \Gamma \end{array}$$

Double Cover of $\mathbf{SL}_2(\mathbb{Z})$

$$\widetilde{\mathrm{SL}_2}(\mathbb{Z}) = \left\{ \left(\gamma, \pm\sqrt{c\tau + d}\right) \mid \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\}$$

is a double cover of $\mathrm{SL}_2(\mathbb{Z})$.

$\widetilde{\mathrm{SL}_2}(\mathbb{Z})$ still acts on $\mathfrak{h}$ via linear fractional transformations.

Vector-Valued Modular Forms with Type

Choose

- A n-dim. rep., $\rho : \widetilde{\mathrm{SL}_2}(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{C})$
- A discrete subgroup, $\widetilde{\Gamma} \subset \widetilde{\mathrm{SL}_2}(\mathbb{Z})$

Definition: A function $F : \mathfrak{h} \to \mathbb{C}^n$ is a modular form on $\widetilde{\Gamma}$ of weight $k \in \frac{1}{2}\mathbb{Z}$ and type ρ if it satifies

$$F(\gamma z) = j(\gamma, z)^k \rho(\gamma) F(z)$$

for every $\gamma \in \widetilde{\Gamma}$.

OR

$$F|_{k,\rho}\gamma = \rho(\gamma)F \quad \text{for} \quad \gamma \in \widetilde{\Gamma}$$

Compare to scalar-valued criterion:

$$f|_{k,\chi}\gamma = \chi(\gamma)f \quad \text{for} \quad \gamma \in \Gamma$$

Examples:

For $n = 1$, $\widetilde{\Gamma} = \widetilde{\Gamma_0(4)}$, $k = \frac{1}{2}$, and $\chi_\theta(\gamma) = \mp i^{(d-1)/2} \left(\frac{c}{d}\right)$

$$\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}$$

For $n > 1$: ???????????

The $\mathbb{C}$-Vector Space

Let Λ be a finite abelian (additive) group with a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form Q_N such that $NQ_N(\lambda) = 0$ for all $\lambda \in \Lambda$.

Example: A lattice L with quadratic form, $\Lambda = L^\vee/L$.

Take $\mathbb{C}^{|\Lambda|}$ with basis $\{e_\lambda\}_{\lambda\in\Lambda}$.

The Character and Representation

We'll need a scalar-valued modular form of weight k and character

$$\chi_\Lambda = \chi_\theta^{2k+\left(\frac{-1}{|\Lambda|}\right)-1} \cdot \left(\frac{d}{2^{2k}|\Lambda|}\right)$$

Example: A product of forms of the type $\Delta^{\frac{j}{24}}(iz)$ for $i, j \in \mathbb{Z}$.

Choose ρ_Λ such that it satisfies

$$\rho_\Lambda(\gamma)e_\lambda = \chi_\Lambda(\gamma)e_{a\lambda}$$

for all $\gamma \in \widetilde{\Gamma_0(N)}$ and all $\lambda \in \Lambda$ with $Q_N(\lambda) = 0$.

Example: Weil Representation

The Construction

For f a modular form on $\Gamma_0(N)$ of weight k and type χ_Λ, define

$$F_f = \sum_{\gamma \in \widetilde{\Gamma_0(N)} \backslash \widetilde{\mathrm{SL}_2}(\mathbb{Z})} (f|_{k,\chi_\Lambda}\gamma)\rho_\Lambda(\gamma^{-1})e_0.$$

Then F_f is a $\mathbb{C}^{|\Lambda|}$-valued modular form on $\widetilde{\Gamma_0(N)}$ of weight k and type ρ_Λ.

Example: $\Lambda = {}^{\frac{1}{2}\mathbb{Z}}/_{\mathbb{Z}} \oplus {}^{\frac{1}{2}\mathbb{Z}}/_{\mathbb{Z}}$ with the quadratic form

$$Q_4((a,b)) = a^2 + b^2.$$

$$f = \frac{\Delta^{\frac{10}{24}}(2z)}{\Delta^{\frac{4}{24}}(z)\Delta^{\frac{4}{24}}(4z)}$$

is weight 1 with the correct character.

Let $f_i = f|_{k,\chi_\Lambda}\gamma_i$. Then

$$F_f = \begin{pmatrix} f_1 - \frac{i}{2}(f_2 - f_3 + f_5 + f_6) \\ \frac{i}{2}(f_6 - f_2) - \frac{1}{2}(f_3 + f_5) \\ \frac{i}{2}(f_6 - f_2) - \frac{1}{2}(f_3 + f_5) \\ -f_4 + \frac{i}{2}(f_5 - f_2 - f_3 - f_6) \end{pmatrix}$$

is a $\mathbb{C}^4$-valued modular form on $\widetilde{\Gamma_0(4)}$ of weight 1 and type ρ_Λ.

Extra Property: F_f is invariant under transformations T that satisfy

$$T(e_\lambda) = e_{\lambda'} \;\Rightarrow\; Q_N(\lambda) = Q_N(\lambda')$$