Alternative Carries for Base-b Addition

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Objectives

This research was done to show the methods and mapping used for different carries in a given number base. This is done by:

- Recalling Daniel Isaksen's model for the 2-digit case.
- Looking at the n-digit case.
- Looking at the unlimited-digit case.

Introduction

We all know how to add base-10 numbers, but what happens when we change the number base? More importantly, what happens when we change the carry for various length strings of numbers? As it turns out, there are isomorphisms that can send an n-digit number with a basic carry to the same digit number with any other carry. We can use these general rules to predict then what maight happen with and unlimited, or even infinite, digit situation.

Important Definitions

The following definitions were instrumental in completeing this research:

1. In base-10,
$345 = 3 \cdot 100 + 4 \cdot 10 + 5 = 3 \cdot 10^2 + 4 \cdot 10^1 + 5 \cdot 10^0$.
2. Consider the base-7 representation $n = 345_7$. Then,
$n = 3 \cdot 7^2 + 4 \cdot 7^1 + 5 \cdot 7^0 = 180$.
3. For $a, b \in \mathbb{Z}$ where $b > 0$, we will let $a \mod b$ denote the remainder of a when divided by b. Also,
$\mathbb{Z}_b = \{0, 1, ..., b-1\}$.

The 2-digit Case

We look at Daniel Isaksen's model for 2-digit addition.
Let $\mathbb{Z}_b^2 = \{[d_1][d_0] : d_i \in \mathbb{Z}_b\}$ be the set of 2-digit base-b representations with d_1 representing the b digit and d_0 representing the ones digit.
For $[c_1][c_0], [d_1][d_0] \in \mathbb{Z}_b^2$, let

$$[c_1][c_0] + [d_1][d_0] = [c_1 + d_1 + z_b(c_0 + d_0)][c_0 + d_0],$$

where

$$z_b(c_0 + d_0) = \left\lfloor \frac{c_0 + d_0}{b} \right\rfloor$$

is the carry that counts how many groups of size b are in $c_0 + d_0$.
We can implement this model using a different carry as well!
If we let k be the new carry when we regroup, then for $[c_1][c_0], [d_1][d_0] \in \mathbb{Z}_b^2$, let

$$[c_1][c_0] +_k [d_1][d_0] = [c_1 + d_1 + kz_b(c_0 + d_0)][c_0 + d_0].$$

2-digit Thereom

Isaksen goes on to develope the follwing Theroem:

If $gcd(b, k) = 1$, then $(\mathbb{Z}_b^2, k) \cong (\mathbb{Z}_b^2, 1) \cong \mathbb{Z}_{b^2}$.

The Isomorphism that maps $(\mathbb{Z}_b^2, 1) \to (\mathbb{Z}_b^2, k)$ is defined as

$$\phi([d_1][d_0]) \to [kd_1][d_0].$$

As an example, let's look at 22_7 as a summand. Implementing the isomorphism, we see that

$$22_7 = [2][2] \to [5 \cdot 2][2] = [3][2] = 32_7.$$

Notice this is a base-7 number with a carry of $k = 5$.

The n-digit Case

Now that we have this model for the 2-digit case, we can talk about the n-digit case.
Let $\mathbb{Z}_b^n = \{[d_n][d_{n-1}]...[d_1][d_0] : d_i \in \mathbb{Z}_b\}$. Define $+_k$ on $\mathbb{Z}_b^n$ by $[c_n][c_{n-1}]...[c_1][c_0] +_k [d_n][d_{n-1}]...[d_1][d_0] = [e_n][e_{n-1}]...[e_1][e_0]$ where $f_i = c_i + d_i + kz_b(f_{i-1})$, $f_{-1} = 0$, and $e_i = f_i \mod b$.
This may look confusing, but notice it is based strictly on the 2-digit model, just for an n-digit string of numbers. Let's look at an example with the addition of two 4-digit numbers:
Compute $3161_7 +_5 1146_7$ with carries of 5.

$$\begin{array}{r} 3\,1\,6\,1_7 \\ +_5 \quad 1\,1\,4\,6_7 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle 5\ \ 10\ 5} \quad \\ 3\,1\,6\,1_7 \\ +_5\ 1\,1\,4\,6_7 \\ \hline 2\,5\,1\,0_7 \end{array}$$

Notice that we stop the carries after the nth digit. This is important leading up to the next section involving unlimitied-digit numbers.

n-digit Theorem

If $gcd(b, k) = 1$, then $(\mathbb{Z}_b^n, k) \cong (\mathbb{Z}_b^n, 1) \cong \mathbb{Z}_{b^n}$ when b is the base and k is the carry.
The Isomorphism that maps $(\mathbb{Z}_b^3, 1) \to (\mathbb{Z}_b^3, k)$ is defined as

$$\phi([d_2][d_1][d_0]) \to \left[k^2 d_2 + k \left\lfloor \frac{kd_1}{b} \right\rfloor\right][kd_1][d_0].$$

This can be expandd for n-digits, but gets a bit messy. It is essentially a string of nested floor functions. The point is made clearly though by simply looking at the 3-digit case. Let's look at an example:
For $b = 7$ and $k = 5$, $234_7 \in (\mathbb{Z}_7^3, 1) = [2][3][4] \to \left[5^2 \cdot 2 + 5 \left\lfloor \frac{5 \cdot 3}{7} \right\rfloor\right][5 \cdot 3][4] = [4][1][4] \in (\mathbb{Z}_7^3, 5)$.

The Unlimited-digit Case

Let $\mathbb{Z}_b^\infty = \{...[d_n][d_{n-1}]...[d_1][d_0] : d_i \in \mathbb{Z}_b\}$.
Let the homomorphism ϕ^{-1} map $(\mathbb{Z}_b^\infty, k) \to (\mathbb{Z}_b^\infty, 1)$. Notice that the map is an inverse of our previous mappings.
So, $\phi^{-1}([d_n][d_{n-1}]...[d_1][d_0]) \to d_n(\frac{b}{k})^n + d_{n-1}(\frac{b}{k})^{n-1} + ... + d_1(\frac{b}{k})^1 + d_0(\frac{b}{k})^0$.

Unlimited digit ?=? ∞-digit

There are really only two options for $(\mathbb{Z}_b^\infty, k)$:
Allowing only finite-length digit strings:

$\Rightarrow$ Not a group (no inverses).

Allowing ∞-digit strings:

$\Rightarrow$ For b prime, $(\mathbb{Z}_b^\infty, k) \cong b$-adic integers.

This turns out to be a group if we allow infinite digit strings, as we can now find inverses (as p-adic numbers). Consider the following:

$$\begin{array}{r} {\scriptstyle \cdots\ 5\ 5} \quad \\ 4_7 \\ +_5\ ...2\,2\,3_7 \\ \hline ...\,0\,0\,0_7 \end{array}$$

Future research includes implementing an isomorphism for a mapping that includes the infinite-digit case.

References

[1]. Daniel Isaksen,"A Comohological Viewpoint on Elementary School Arithmetic"

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