Generalized Factorials and Taylor Expansions

Michael R. Pilla
Winona State University

April 16, 2010

Factorials

- Recall the definition of a factorial

$$n! = n \cdot (n-1) \cdots 2 \cdot 1 =$$

Factorials

- Recall the definition of a factorial

$$n! = n \cdot (n-1) \cdots 2 \cdot 1 = (n-0)(n-1)(n-2) \cdots (n-(n-2))(n-(n-1)).$$

Factorials

- Recall the definition of a factorial

$$n! = n\cdot(n-1)\cdots 2\cdot 1 = (n-0)(n-1)(n-2)\cdots(n-(n-2))(n-(n-1)).$$

- For $S = \{a_0, a_1, a_2, a_3, ...\} \subseteq \mathbb{N}$,

  Define $n!_S = (a_n - a_0)(a_n - a_1)\cdots(a_n - a_{n-1})$.

Factorials

- Recall the definition of a factorial

  $$n! = n\cdot(n-1)\cdots 2\cdot 1 = (n-0)(n-1)(n-2)\cdots(n-(n-2))(n-(n-1)).$$

- For $S = \{a_0, a_1, a_2, a_3, \ldots\} \subseteq \mathbb{N}$,

  Define $n!_S = (a_n - a_0)(a_n - a_1)\cdots(a_n - a_{n-1})$.

Pros:

Factorials

- Recall the definition of a factorial

$$n! = n{\cdot}(n{-}1){\cdots}2{\cdot}1 = (n{-}0)(n{-}1)(n{-}2){\cdots}(n{-}(n{-}2))(n{-}(n{-}1)).$$

- For $S = \{a_0, a_1, a_2, a_3, ...\} \subseteq \mathbb{N}$,

  Define $n!_S = (a_n - a_0)(a_n - a_1) \cdots (a_n - a_{n-1})$.

Pros:

- We have a nice formula.

Factorials

- Recall the definition of a factorial

$$n! = n{\cdot}(n{-}1){\cdots}2{\cdot}1 = (n{-}0)(n{-}1)(n{-}2){\cdots}(n{-}(n{-}2))(n{-}(n{-}1)).$$

- For $S = \{a_0, a_1, a_2, a_3, ...\} \subseteq \mathbb{N}$,

  Define $n!_S = (a_n - a_0)(a_n - a_1) \cdots (a_n - a_{n-1})$.

Pros:

- We have a nice formula.
- It works on subsets of $\mathbb{N}$.

Factorials

- Recall the definition of a factorial

$$n! = n\cdot(n-1)\cdots 2\cdot 1 = (n-0)(n-1)(n-2)\cdots(n-(n-2))(n-(n-1)).$$

- For $S = \{a_0, a_1, a_2, a_3, \ldots\} \subseteq \mathbb{N}$,

  Define $n!_S = (a_n - a_0)(a_n - a_1)\cdots(a_n - a_{n-1})$.

Pros:

- We have a nice formula.
- It works on subsets of $\mathbb{N}$.

Cons:

Factorials

- Recall the definition of a factorial

$$n! = n\cdot(n-1)\cdots 2\cdot 1 = (n-0)(n-1)(n-2)\cdots(n-(n-2))(n-(n-1)).$$

- For $S = \{a_0, a_1, a_2, a_3, \ldots\} \subseteq \mathbb{N}$,

  Define $n!_S = (a_n - a_0)(a_n - a_1)\cdots(a_n - a_{n-1})$.

Pros:

- We have a nice formula.
- It works on subsets of $\mathbb{N}$.

Cons:

- It is not theoretically useful.

Factorials

- Recall the definition of a factorial

$$n! = n\cdot(n-1)\cdots 2\cdot 1 = (n-0)(n-1)(n-2)\cdots(n-(n-2))(n-(n-1)).$$

- For $S = \{a_0, a_1, a_2, a_3, ...\} \subseteq \mathbb{N}$,

  Define $n!_S = (a_n - a_0)(a_n - a_1)\cdots(a_n - a_{n-1})$.

Pros:

- We have a nice formula.
- It works on subsets of $\mathbb{N}$.

Cons:

- It is not theoretically useful.
- What about the order of the subset?

Generalized Factorials

- Bhargava: Let's look at prime factorizations and play a game called p-ordering for each prime p.

Generalized Factorials

- Bhargava: Let's look at prime factorizations and play a game called p-ordering for each prime p.
- If the ordering works for all primes simultaneosly, then we can achieve nice formulas.

Generalized Factorials

- Bhargava: Let's look at prime factorizations and play a game called p-ordering for each prime p.
- If the ordering works for all primes simultaneosly, then we can achieve nice formulas.

Theorem

If $S = \{a_i\}$ is p-ordered for all primes simultaneously then

$$n!_S = \left|(a_n - a_0)(a_n - a_1)\cdots(a_n - a_{n-1})\right|.$$

Generalized Factorials

- Bhargava: Let's look at prime factorizations and play a game called p-ordering for each prime p.
- If the ordering works for all primes simultaneosly, then we can achieve nice formulas.

Theorem

If $S = \{a_i\}$ is p-ordered for all primes simultaneously then

$$n!_S = |(a_n - a_0)(a_n - a_1)\cdots(a_n - a_{n-1})|.$$

- If a subset cannot be simultaneously ordered, the formulas are ugly (if and when they exist) and the factorials difficult to calculate.

Generalized Factorials

- Bhargava: Let's look at prime factorizations and play a game called *p*-ordering for each prime *p*.
- If the ordering works for all primes simultaneosly, then we can achieve nice formulas.

Theorem

If $S = \{a_i\}$ is p-ordered for all primes simultaneously then

$$n!_S = |(a_n - a_0)(a_n - a_1) \cdots (a_n - a_{n-1})|.$$

- If a subset cannot be simultaneously ordered, the formulas are ugly (if and when they exist) and the factorials difficult to calculate.
- Goal: Extend factorials to "nice", "natural" subsets of $\mathbb{N}$ that have closed formulas.

Arithmetic Sets

Example

- Consider the arithmetic set $A = \{2, 9, 16, 23, \ldots\}$ (i.e. all integers 2 mod 7). This is p-ordered for all primes simultaneously. Thus

Arithmetic Sets

Example

- Consider the arithmetic set $A = \{2, 9, 16, 23, \ldots\}$ (i.e. all integers 2 mod 7). This is p-ordered for all primes simultaneously. Thus
- $1!_A = (9 - 2) = 7$

Arithmetic Sets

Example

- Consider the arithmetic set $A = \{2, 9, 16, 23, ...\}$ (i.e. all integers 2 mod 7). This is p-ordered for all primes simultaneously. Thus
- $1!_A = (9 - 2) = 7$
- $2!_A = (16 - 2)(16 - 9) = 98$

Arithmetic Sets

Example

- Consider the arithmetic set $A = \{2, 9, 16, 23, ...\}$ (i.e. all integers 2 mod 7). This is p-ordered for all primes simultaneously. Thus
- $1!_A = (9 - 2) = 7$
- $2!_A = (16 - 2)(16 - 9) = 98$
- $3!_A = (23 - 2)(23 - 9)(23 - 16) = 2058$

Arithmetic Sets

Example

- Consider the arithmetic set $A = \{2, 9, 16, 23, ...\}$ (i.e. all integers 2 mod 7). This is p-ordered for all primes simultaneously. Thus
- $1!_A = (9-2) = 7 = 7^1 1!$
- $2!_A = (16-2)(16-9) = 98 = 7^2 2!$
- $3!_A = (23-2)(23-9)(23-16) = 2058 = 7^3 3!$
- $n!_A = 7^n n!$

Arithmetic Sets

Example

- Consider the arithmetic set $A = \{2, 9, 16, 23, ...\}$ (i.e. all integers 2 mod 7). This is p-ordered for all primes simultaneously. Thus
- $1!_A = (9 - 2) = 7 = 7^1 1!$
- $2!_A = (16 - 2)(16 - 9) = 98 = 7^2 2!$
- $3!_A = (23 - 2)(23 - 9)(23 - 16) = 2058 = 7^3 3!$
- $n!_A = 7^n n!$

Example

Let $S = a\mathbb{N} + b$ of all integers b mod a. The natural ordering is p-ordered for all primes simultaneously. Thus

$$n!_{a\mathbb{N}+b} = a^n n!$$

Set of Squares

Example

- The set $\mathbb{Z}^2 = \{0, 1, 4, 9, 16, ...\}$ of square numbers admits a simultaneous p-ordering. Thus

Set of Squares

Example

- The set $\mathbb{Z}^2 = \{0, 1, 4, 9, 16, ...\}$ of square numbers admits a simultaneous *p*-ordering. Thus
- $1!_{\mathbb{Z}^2} = (1 - 0) = 1$

Set of Squares

Example

- The set $\mathbb{Z}^2 = \{0, 1, 4, 9, 16, ...\}$ of square numbers admits a simultaneous p-ordering. Thus
- $1!_{\mathbb{Z}^2} = (1 - 0) = 1$
- $2!_{\mathbb{Z}^2} = (4 - 0)(4 - 1) = 12$

Set of Squares

Example

- The set $\mathbb{Z}^2 = \{0, 1, 4, 9, 16, ...\}$ of square numbers admits a simultaneous p-ordering. Thus
- $1!_{\mathbb{Z}^2} = (1 - 0) = 1$
- $2!_{\mathbb{Z}^2} = (4 - 0)(4 - 1) = 12$
- $3!_{\mathbb{Z}^2} = (9 - 0)(9 - 1)(9 - 4) = 360$

Set of Squares

Example

- The set $\mathbb{Z}^2 = \{0, 1, 4, 9, 16, \ldots\}$ of square numbers admits a simultaneous *p*-ordering. Thus
- $1!_{\mathbb{Z}^2} = (1-0) = 1$
- $2!_{\mathbb{Z}^2} = (4-0)(4-1) = 12$
- $3!_{\mathbb{Z}^2} = (9-0)(9-1)(9-4) = 360$
- $4!_{\mathbb{Z}^2} = (16-0)(16-1)(16-4)(16-9) = 20160$

Set of Squares

Example

- The set $\mathbb{Z}^2 = \{0, 1, 4, 9, 16, ...\}$ of square numbers admits a simultaneous *p*-ordering. Thus
- $1!_{\mathbb{Z}^2} = (1-0) = 1$
- $2!_{\mathbb{Z}^2} = (4-0)(4-1) = 12$
- $3!_{\mathbb{Z}^2} = (9-0)(9-1)(9-4) = 360$
- $4!_{\mathbb{Z}^2} = (16-0)(16-1)(16-4)(16-9) = 20160$

$$n!_{\mathbb{Z}^2} = (n^2-0)(n^2-1)(n^2-4)\cdots(n^2-(n-1)^2)$$

Set of Squares

Example

- The set $\mathbb{Z}^2 = \{0, 1, 4, 9, 16, ...\}$ of square numbers admits a simultaneous p-ordering. Thus
- $1!_{\mathbb{Z}^2} = (1-0) = 1$
- $2!_{\mathbb{Z}^2} = (4-0)(4-1) = 12$
- $3!_{\mathbb{Z}^2} = (9-0)(9-1)(9-4) = 360$
- $4!_{\mathbb{Z}^2} = (16-0)(16-1)(16-4)(16-9) = 20160$

$$\begin{aligned} n!_{\mathbb{Z}^2} &= (n^2-0)(n^2-1)(n^2-4)\cdots(n^2-(n-1)^2) \\ &= (n-0)(n+0)(n-1)(n+1)...(n-(n-1))(n+(n-1)) \\ &= \frac{2n}{2}(2n-1)(2n-2)\cdots(n)(n-1)\cdots(1) \\ &= \frac{(2n)!}{2} \end{aligned}$$

Twice Triangulars (Squares Modified)

Example

- Likewise, one can show the set $2\mathbb{T} = \{n^2 + n \mid n \in \mathbb{N}\} = \{0, 2, 6, 12, 20, \ldots\}$ admits a simultaneous p-ordering. Thus

Twice Triangulars (Squares Modified)

Example

- Likewise, one can show the set $2\mathbb{T} = \{n^2 + n \mid n \in \mathbb{N}\} = \{0, 2, 6, 12, 20, \ldots\}$ admits a simultaneous p-ordering. Thus
- $1!_{2\mathbb{T}} = (2 - 0) = 2$

Twice Triangulars (Squares Modified)

Example

- Likewise, one can show the set $2\mathbb{T} = \{n^2 + n \mid n \in \mathbb{N}\} = \{0, 2, 6, 12, 20, \dots\}$ admits a simultaneous p-ordering. Thus
- $1!_{2\mathbb{T}} = (2 - 0) = 2$
- $2!_{2\mathbb{T}} = (6 - 0)(6 - 2) = 24$

Twice Triangulars (Squares Modified)

Example

- Likewise, one can show the set $2\mathbb{T} = \{n^2 + n \mid n \in \mathbb{N}\} = \{0, 2, 6, 12, 20, \ldots\}$ admits a simultaneous p-ordering. Thus
- $1!_{2\mathbb{T}} = (2 - 0) = 2$
- $2!_{2\mathbb{T}} = (6 - 0)(6 - 2) = 24$
- $3!_{2\mathbb{T}} = (12 - 0)(12 - 2)(12 - 6) = 720$

Twice Triangulars (Squares Modified)

Example

- Likewise, one can show the set $2\mathbb{T} = \{n^2 + n \mid n \in \mathbb{N}\} = \{0, 2, 6, 12, 20, \ldots\}$ admits a simultaneous p-ordering. Thus
- $1!_{2\mathbb{T}} = (2 - 0) = 2$
- $2!_{2\mathbb{T}} = (6 - 0)(6 - 2) = 24$
- $3!_{2\mathbb{T}} = (12 - 0)(12 - 2)(12 - 6) = 720$
- $4!_{2\mathbb{T}} = (20 - 0)(20 - 2)(20 - 6)(20 - 12) = 40320$

Twice Triangulars (Squares Modified)

Example

- Likewise, one can show the set $2\mathbb{T} = \{n^2 + n \mid n \in \mathbb{N}\} = \{0, 2, 6, 12, 20, \ldots\}$ admits a simultaneous p-ordering. Thus
- $1!_{2\mathbb{T}} = (2 - 0) = 2$
- $2!_{2\mathbb{T}} = (6 - 0)(6 - 2) = 24$
- $3!_{2\mathbb{T}} = (12 - 0)(12 - 2)(12 - 6) = 720$
- $4!_{2\mathbb{T}} = (20 - 0)(20 - 2)(20 - 6)(20 - 12) = 40320$
- $n!_{2\mathbb{T}} = (2n)!$

A Geometric Progression

Example

- Consider the geometric progression $G = \{5 \cdot 3^n\} = \{5, 15, 45, 135, 405, ...\}$. This set admits a simultaneous p-ordering and thus

A Geometric Progression

Example

- Consider the geometric progression $G = \{5 \cdot 3^n\} = \{5, 15, 45, 135, 405, \ldots\}$. This set admits a simultaneous p-ordering and thus
- $1!_G = (15 - 5) = 10$

A Geometric Progression

Example

- Consider the geometric progression $G = \{5 \cdot 3^n\} = \{5, 15, 45, 135, 405, \ldots\}$. This set admits a simultaneous p-ordering and thus
- $1!_G = (15 - 5) = 10$
- $2!_G = (45 - 5)(45 - 15) = 1200$

A Geometric Progression

Example

- Consider the geometric progression $G = \{5 \cdot 3^n\} = \{5, 15, 45, 135, 405, \ldots\}$. This set admits a simultaneous *p*-ordering and thus
- $1!_G = (15 - 5) = 10$
- $2!_G = (45 - 5)(45 - 15) = 1200$
- $3!_G = (135 - 5)(135 - 15)(135 - 45) = 1404000$

A Geometric Progression

Example

- Consider the geometric progression $G = \{5 \cdot 3^n\} = \{5, 15, 45, 135, 405, \ldots\}$. This set admits a simultaneous p-ordering and thus
- $1!_G = (15 - 5) = 10$
- $2!_G = (45 - 5)(45 - 15) = 1200$
- $3!_G = (135 - 5)(135 - 15)(135 - 45) = 1404000$
- $4!_G = (405-5)(405-15)(405-45)(405-135) = 15163200000$

A General Geometric Progression

Example

The geometric progression $aq^{\mathbb{N}}$ gives a simultaneous p-ordering. Thus

A General Geometric Progression

Example

The geometric progression $aq^{\mathbb{N}}$ gives a simultaneous p-ordering. Thus

$$n!_{aq^{\mathbb{N}}} = (aq^n - a)(aq^n - aq)\cdots(aq^n - aq^{n-1})$$

A General Geometric Progression

Example

The geometric progression $aq^{\mathbb{N}}$ gives a simultaneous p-ordering. Thus

$$\begin{aligned} n!_{aq^{\mathbb{N}}} &= (aq^n - a)(aq^n - aq)\cdots(aq^n - aq^{n-1}) \\ &= a^n q^n (1-q^{-n})(1-q^{1-n})\cdots(1-q^{-2})(1-q^{-1})) \end{aligned}$$

A General Geometric Progression

Example

The geometric progression $aq^{\mathbb{N}}$ gives a simultaneous p-ordering. Thus

$$\begin{aligned} n!_{aq^{\mathbb{N}}} &= (aq^n - a)(aq^n - aq)\cdots(aq^n - aq^{n-1}) \\ &= a^n q^n (1 - q^{-n})(1 - q^{1-n})\cdots(1 - q^{-2})(1 - q^{-1})) \\ &= (-aq)^n q^{\frac{-n(n+1)}{2}}(1 - q)(1 - q^2)\cdots(1 - q^n) \end{aligned}$$

A General Geometric Progression

Example

The geometric progression $aq^{\mathbb{N}}$ gives a simultaneous p-ordering. Thus

$$
\begin{aligned}
n!_{aq^{\mathbb{N}}} &= (aq^n - a)(aq^n - aq)\cdots(aq^n - aq^{n-1}) \\
&= a^n q^n (1 - q^{-n})(1 - q^{1-n})\cdots(1 - q^{-2})(1 - q^{-1})) \\
&= (-aq)^n q^{\frac{-n(n+1)}{2}}(1 - q)(1 - q^2)\cdots(1 - q^n) \\
&= (-aq)^n q^{\frac{-n(n+1)}{2}}(q : q)_n
\end{aligned}
$$

where $(q : q)_n$ is the q-Pochhammer symbol.

Non-Simultaneous Sets

Example

- While it can be shown that the primes do NOT admit a simultaneous p-ordering, Bhargava gives an intricate formula to calculate them.

Non-Simultaneous Sets

Example

- While it can be shown that the primes do NOT admit a simultaneous p-ordering, Bhargava gives an intricate formula to calculate them.

$$n!_{\mathbb{P}} = \prod_p p^{\lfloor \frac{n-1}{p-1} \rfloor + \lfloor \frac{n-1}{p(p-1)} \rfloor + \lfloor \frac{n-1}{p^2(p-1)} \rfloor + \cdots}$$

Non-Simultaneous Sets

Example

- While it can be shown that the primes do NOT admit a simultaneous p-ordering, Bhargava gives an intricate formula to calculate them.

$$n!_{\mathbb{P}} = \prod_{p} p^{\lfloor \frac{n-1}{p-1} \rfloor + \lfloor \frac{n-1}{p(p-1)} \rfloor + \lfloor \frac{n-1}{p^2(p-1)} \rfloor + \cdots}$$

- $\{n!_{\mathbb{P}}\} = \{1, 2, 24, 48, 5670, \ldots\}$

Non-Simultaneous Sets

Example

- While it can be shown that the primes do NOT admit a simultaneous p-ordering, Bhargava gives an intricate formula to calculate them.

$$n!_{\mathbb{P}} = \prod_p p^{\lfloor \frac{n-1}{p-1} \rfloor + \lfloor \frac{n-1}{p(p-1)} \rfloor + \lfloor \frac{n-1}{p^2(p-1)} \rfloor + \cdots}$$

- $\{n!_{\mathbb{P}}\} = \{1, 2, 24, 48, 5670, \ldots\}$

Example

- The set of cubes $\mathbb{N}^3$ does not admit a simultaneous p-ordering and has no nice formula.

Non-Simultaneous Sets

Example

- While it can be shown that the primes do NOT admit a simultaneous p-ordering, Bhargava gives an intricate formula to calculate them.

$$n!_{\mathbb{P}} = \prod_p p^{\lfloor \frac{n-1}{p-1} \rfloor + \lfloor \frac{n-1}{p(p-1)} \rfloor + \lfloor \frac{n-1}{p^2(p-1)} \rfloor + \cdots}$$

- $\{n!_{\mathbb{P}}\} = \{1, 2, 24, 48, 5670, \ldots\}$

Example

- The set of cubes $\mathbb{N}^3$ does not admit a simultaneous p-ordering and has no nice formula.
- $\{n!_{\mathbb{N}^3}\} = \{1, 2, 504, 504, 35280, \ldots\}$

Summary of Examples

$$\mathbb{N} \quad \leftrightsquigarrow \quad n!_{\mathbb{N}} = n!$$

Summary of Examples

$$\mathbb{N} \quad \leftrightsquigarrow \quad n!_{\mathbb{N}} = n!$$

$$a\mathbb{N} + b \quad \leftrightsquigarrow \quad n!_{a\mathbb{N}+b} = a^n n!$$

Summary of Examples

$$\mathbb{N} \quad \leftrightsquigarrow \quad n!_{\mathbb{N}} = n!$$

$$a\mathbb{N} + b \quad \leftrightsquigarrow \quad n!_{a\mathbb{N}+b} = a^n n!$$

$$2\mathbb{T} \quad \leftrightsquigarrow \quad n!_{2\mathbb{T}} = (2n)!$$

Summary of Examples

$$\mathbb{N} \quad \leftrightsquigarrow \quad n!_{\mathbb{N}} = n!$$

$$a\mathbb{N} + b \quad \leftrightsquigarrow \quad n!_{a\mathbb{N}+b} = a^n n!$$

$$2\mathbb{T} \quad \leftrightsquigarrow \quad n!_{2\mathbb{T}} = (2n)!$$

$$\mathbb{Z}^2 \quad \leftrightsquigarrow \quad n!_{\mathbb{Z}^2} = \frac{(2n)!}{2}$$

Summary of Examples

$$\mathbb{N} \quad \leftrightsquigarrow \quad n!_{\mathbb{N}} = n!$$

$$a\mathbb{N} + b \quad \leftrightsquigarrow \quad n!_{a\mathbb{N}+b} = a^n n!$$

$$2\mathbb{T} \quad \leftrightsquigarrow \quad n!_{2\mathbb{T}} = (2n)!$$

$$\mathbb{Z}^2 \quad \leftrightsquigarrow \quad n!_{\mathbb{Z}^2} = \frac{(2n)!}{2}$$

$$aq^{\mathbb{N}} \quad \leftrightsquigarrow \quad n!_{aq^{\mathbb{N}}} = (-aq)^n q^{\frac{-n(n+1)}{2}} (q : q)_n$$

Summary of Examples

$$\mathbb{N} \quad \leftrightsquigarrow \quad n!_{\mathbb{N}} = n!$$

$$a\mathbb{N} + b \quad \leftrightsquigarrow \quad n!_{a\mathbb{N}+b} = a^n n!$$

$$2\mathbb{T} \quad \leftrightsquigarrow \quad n!_{2\mathbb{T}} = (2n)!$$

$$\mathbb{Z}^2 \quad \leftrightsquigarrow \quad n!_{\mathbb{Z}^2} = \frac{(2n)!}{2}$$

$$aq^{\mathbb{N}} \quad \leftrightsquigarrow \quad n!_{aq^{\mathbb{N}}} = (-aq)^n q^{\frac{-n(n+1)}{2}} (q : q)_n$$

$$\mathbb{P} \quad \leftrightsquigarrow \quad n!_{\mathbb{P}} = \prod_p p^{(stuff)}$$

Question

Question

Recall that one can write e^x as a Taylor series.

Question

Recall that one can write e^x as a Taylor series. That is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Question

Recall that one can write e^x as a Taylor series. That is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Question: What is the $!_S$-analogue to this equation?

Question

Recall that one can write e^x as a Taylor series. That is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Question: What is the $!_S$-analogue to this equation?

As it turns out, this is the wrong question to ask.

Right Question

Raising to the power of m, we can write $(e^x)^m$ as follows:

Right Question

Raising to the power of m, we can write $(e^x)^m$ as follows:

$$(e^x)^m = e^{mx} = \sum_{n=0}^{\infty} \frac{m^n}{n!} x^n = 1 + \frac{m}{1!}x + \frac{m^2}{2!}x^2 + \frac{m^3}{3!}x^3 + \cdots$$

Right Question

Raising to the power of m, we can write $(e^x)^m$ as follows:

$$(e^x)^m = e^{mx} = \sum_{n=0}^{\infty} \frac{m^n}{n!} x^n = 1 + \frac{m}{1!}x + \frac{m^2}{2!}x^2 + \frac{m^3}{3!}x^3 + \cdots$$

Right Question: What is the $!_S$-analogue of this equation?

Right Question

Raising to the power of m, we can write $(e^x)^m$ as follows:

$$(e^x)^m = e^{mx} = \sum_{n=0}^{\infty} \frac{m^n}{n!} x^n = 1 + \frac{m}{1!} x + \frac{m^2}{2!} x^2 + \frac{m^3}{3!} x^3 + \cdots$$

Right Question: What is the $!_S$-analogue of this equation?

Goal:

- 1) The numerator of each "coefficient" is a polynomial in m.
- 2) The denominator of each "coefficient" is a factorial.

$(a\mathbb{N}+b)$ -analogue

Example

$\left(\frac{a}{a-x}\right)^m = \left(1-\frac{x}{a}\right)^{-m}$

$(a\mathbb{N}+b)$ -analogue

Example

$$\left(\frac{a}{a-x}\right)^m = \left(1-\frac{x}{a}\right)^{-m}$$

$$= 1 + \frac{m}{a}x + \frac{m(m-1)}{2a^2}x^2 + \frac{m(m-1)(m-2)}{6a^3}x^3$$

$$+ \frac{m(m-1)(m-2)(m-3)}{24a^4}x^4 + \frac{m(m-1)(m-2)(m-3)(m-4)}{120a^5}x^5 + \cdots$$

$(a\mathbb{N}+b)$ -analogue

Example

$$\left(\frac{a}{a-x}\right)^m = \left(1-\frac{x}{a}\right)^{-m}$$
$$= 1 + \frac{m}{a}x + \frac{m(m-1)}{2a^2}x^2 + \frac{m(m-1)(m-2)}{6a^3}x^3$$
$$+ \frac{m(m-1)(m-2)(m-3)}{24a^4}x^4 + \frac{m(m-1)(m-2)(m-3)(m-4)}{120a^5}x^5 + \cdots$$

Notice our numerators are polynomials in m and our denominators are $a^n n! = n!_{a\mathbb{N}+b}$.

$(a\mathbb{N}+b)$ -analogue

Example

$$\left(\frac{a}{a-x}\right)^m = \left(1-\frac{x}{a}\right)^{-m}$$

$$= 1 + \frac{m}{a}x + \frac{m(m-1)}{2a^2}x^2 + \frac{m(m-1)(m-2)}{6a^3}x^3$$

$$+ \frac{m(m-1)(m-2)(m-3)}{24a^4}x^4 + \frac{m(m-1)(m-2)(m-3)(m-4)}{120a^5}x^5 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{P_{a\mathbb{N}+b,n}(m)}{n!_{a\mathbb{N}+b}}x^n$$

Notice our numerators are polynomials in m and our denominators are $a^n n! = n!_{a\mathbb{N}+b}$.

$2\mathbb{T}$ -analogue

Example

$cos^m(\sqrt{x}) =$

$2\mathbb{T}$ -analogue

Example

$$cos^m(\sqrt{x}) = 1 - \frac{m}{2}x + \frac{m+3m(m-1)}{24}x^2$$
$$-\frac{15m(m-1)+m+15m(m-1)(m-2)}{720}x^3 + \cdots$$

$2\mathbb{T}$ -analogue

Example

$$\cos^m(\sqrt{x}) = 1 - \frac{m}{2}x + \frac{m+3m(m-1)}{24}x^2$$

$$-\frac{15m(m-1)+m+15m(m-1)(m-2)}{720}x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{P_{2\mathbb{T},n}(m)}{n!_{2\mathbb{T}}}x^n$$

$2\mathbb{T}$ -analogue

Example

$$cos^m(\sqrt{x}) = 1 - \tfrac{m}{2}x + \tfrac{m+3m(m-1)}{24}x^2 - \tfrac{15m(m-1)+m+15m(m-1)(m-2)}{720}x^3 + \cdots = \textstyle\sum_{n=0}^{\infty} \frac{P_{2\mathbb{T},n}(m)}{n!_{2\mathbb{T}}}x^n$$

Note that by allowing multiplication by scalars, the $\mathbb{Z}^2$-analogue is

$$2cos^m(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{P_{\mathbb{Z}^2,n}(m)}{n!_{\mathbb{Z}^2}}x^n.$$

$\mathbb{P}$ -analogue

Example

$$\left(-\frac{ln(1-x)}{x}\right)^m = 1 + \frac{m}{2}x + \frac{m(3m+5)}{24}x^2 + \frac{m(m^2+5m+6)}{48}x^3 + \cdots$$

$\mathbb{P}$ -analogue

Example

$$\left(-\frac{ln(1-x)}{x}\right)^m = 1 + \frac{m}{2}x + \frac{m(3m+5)}{24}x^2 + \frac{m(m^2+5m+6)}{48}x^3 + \cdots$$

- Notice that the coefficients in the denominator seem to be the same as the factorials for the set of primes.

$\mathbb{P}$ -analogue

Example

$$\left(-\frac{ln(1-x)}{x}\right)^m = 1+\frac{m}{2}x+\frac{m(3m+5)}{24}x^2+\frac{m(m^2+5m+6)}{48}x^3+\cdots$$
$$= \sum_{n=0}^{\infty}\frac{P_{\mathbb{P},n}(m)}{n!_{\mathbb{P}}}x^n$$

- Notice that the coefficients in the denominator seem to be the same as the factorials for the set of primes.
- It turns out that this is so (Chabert, 2005).

Summary of $!_S$-analogues

$$\mathbb{N} \quad \leftrightsquigarrow \quad n!_{\mathbb{N}} = n! \quad \leftrightsquigarrow$$

$$a\mathbb{N} + b \quad \leftrightsquigarrow \quad n!_{a\mathbb{N}+b} = a^n n! \quad \leftrightsquigarrow$$

$$2\mathbb{T} \quad \leftrightsquigarrow \quad n!_{2\mathbb{T}} = (2n)! \quad \leftrightsquigarrow$$

$$\mathbb{Z}^2 \quad \leftrightsquigarrow \quad n!_{\mathbb{Z}^2} = \frac{(2n)!}{2} \quad \leftrightsquigarrow$$

$$aq^{\mathbb{N}} \quad \leftrightsquigarrow \quad n!_{aq^{\mathbb{N}}} = (-aq)^n q^{\frac{-n(n+1)}{2}} (q:q)_n \quad \leftrightsquigarrow$$

$$\mathbb{P} \quad \leftrightsquigarrow \quad n!_{\mathbb{P}} = \prod_p p^{(stuff)} \quad \leftrightsquigarrow$$

Summary of $!_S$-analogues

$\mathbb{N}$	$\leftrightsquigarrow$	$n!_{\mathbb{N}} = n!$	$\leftrightsquigarrow$	$(e^x)^m$
$a\mathbb{N}+b$	$\leftrightsquigarrow$	$n!_{a\mathbb{N}+b} = a^n n!$	$\leftrightsquigarrow$	
$2\mathbb{T}$	$\leftrightsquigarrow$	$n!_{2\mathbb{T}} = (2n)!$	$\leftrightsquigarrow$	
$\mathbb{Z}^2$	$\leftrightsquigarrow$	$n!_{\mathbb{Z}^2} = \frac{(2n)!}{2}$	$\leftrightsquigarrow$	
$aq^{\mathbb{N}}$	$\leftrightsquigarrow$	$n!_{aq^{\mathbb{N}}} = (-aq)^n q^{\frac{-n(n+1)}{2}} (q:q)_n$	$\leftrightsquigarrow$	
$\mathbb{P}$	$\leftrightsquigarrow$	$n!_{\mathbb{P}} = \prod_p p^{(stuff)}$	$\leftrightsquigarrow$	

Summary of $!_S$-analogues

$$\mathbb{N} \quad \leftrightsquigarrow \quad n!_{\mathbb{N}} = n! \quad \leftrightsquigarrow \quad (e^x)^m$$

$$a\mathbb{N} + b \quad \leftrightsquigarrow \quad n!_{a\mathbb{N}+b} = a^n n! \quad \leftrightsquigarrow \quad \left(\frac{a}{a-x}\right)^m$$

$$2\mathbb{T} \quad \leftrightsquigarrow \quad n!_{2\mathbb{T}} = (2n)! \quad \leftrightsquigarrow$$

$$\mathbb{Z}^2 \quad \leftrightsquigarrow \quad n!_{\mathbb{Z}^2} = \frac{(2n)!}{2} \quad \leftrightsquigarrow$$

$$aq^{\mathbb{N}} \quad \leftrightsquigarrow \quad n!_{aq^{\mathbb{N}}} = (-aq)^n q^{\frac{-n(n+1)}{2}} (q:q)_n \quad \leftrightsquigarrow$$

$$\mathbb{P} \quad \leftrightsquigarrow \quad n!_{\mathbb{P}} = \prod_p p^{(stuff)} \quad \leftrightsquigarrow$$

Summary of $!_S$-analogues

$\mathbb{N}$	$\leftrightsquigarrow$	$n!_{\mathbb{N}} = n!$	$\leftrightsquigarrow$	$(e^x)^m$
$a\mathbb{N}+b$	$\leftrightsquigarrow$	$n!_{a\mathbb{N}+b} = a^n n!$	$\leftrightsquigarrow$	$\left(\frac{a}{a-x}\right)^m$
$2\mathbb{T}$	$\leftrightsquigarrow$	$n!_{2\mathbb{T}} = (2n)!$	$\leftrightsquigarrow$	$\cos^m(\sqrt{x})$
$\mathbb{Z}^2$	$\leftrightsquigarrow$	$n!_{\mathbb{Z}^2} = \frac{(2n)!}{2}$	$\leftrightsquigarrow$	
$aq^{\mathbb{N}}$	$\leftrightsquigarrow$	$n!_{aq^{\mathbb{N}}} = (-aq)^n q^{\frac{-n(n+1)}{2}} (q:q)_n$	$\leftrightsquigarrow$	
$\mathbb{P}$	$\leftrightsquigarrow$	$n!_{\mathbb{P}} = \prod_p p^{(stuff)}$	$\leftrightsquigarrow$	

Summary of $!_S$-analogues

$\mathbb{N}$	$\leftrightsquigarrow$	$n!_{\mathbb{N}} = n!$	$\leftrightsquigarrow$	$(e^x)^m$
$a\mathbb{N}+b$	$\leftrightsquigarrow$	$n!_{a\mathbb{N}+b} = a^n n!$	$\leftrightsquigarrow$	$(\frac{a}{a-x})^m$
$2\mathbb{T}$	$\leftrightsquigarrow$	$n!_{2\mathbb{T}} = (2n)!$	$\leftrightsquigarrow$	$cos^m(\sqrt{x})$
$\mathbb{Z}^2$	$\leftrightsquigarrow$	$n!_{\mathbb{Z}^2} = \frac{(2n)!}{2}$	$\leftrightsquigarrow$	$2cos^m(\sqrt{x})$
$aq^{\mathbb{N}}$	$\leftrightsquigarrow$	$n!_{aq^{\mathbb{N}}} = (-aq)^n q^{\frac{-n(n+1)}{2}}(q:q)_n$	$\leftrightsquigarrow$	
$\mathbb{P}$	$\leftrightsquigarrow$	$n!_{\mathbb{P}} = \prod_p p^{(stuff)}$	$\leftrightsquigarrow$	

Summary of $!_S$-analogues

$\mathbb{N}$	$\leftrightsquigarrow$	$n!_{\mathbb{N}} = n!$	$\leftrightsquigarrow$	$(e^x)^m$
$a\mathbb{N} + b$	$\leftrightsquigarrow$	$n!_{a\mathbb{N}+b} = a^n n!$	$\leftrightsquigarrow$	$\left(\frac{a}{a-x}\right)^m$
$2\mathbb{T}$	$\leftrightsquigarrow$	$n!_{2\mathbb{T}} = (2n)!$	$\leftrightsquigarrow$	$cos^m(\sqrt{x})$
$\mathbb{Z}^2$	$\leftrightsquigarrow$	$n!_{\mathbb{Z}^2} = \frac{(2n)!}{2}$	$\leftrightsquigarrow$	$2cos^m(\sqrt{x})$
$aq^{\mathbb{N}}$	$\leftrightsquigarrow$	$n!_{aq^{\mathbb{N}}} = (-aq)^n q^{\frac{-n(n+1)}{2}} (q:q)_n$	$\leftrightsquigarrow$	
$\mathbb{P}$	$\leftrightsquigarrow$	$n!_{\mathbb{P}} = \prod_p p^{(stuff)}$	$\leftrightsquigarrow$	$\left(-\frac{ln(1-x)}{x}\right)^m$

Summary of $!_S$-analogues

$\mathbb{N}$	$\leftrightsquigarrow$	$n!_{\mathbb{N}} = n!$	$\leftrightsquigarrow$	$(e^x)^m$
$a\mathbb{N}+b$	$\leftrightsquigarrow$	$n!_{a\mathbb{N}+b} = a^n n!$	$\leftrightsquigarrow$	$(\frac{a}{a-x})^m$
$2\mathbb{T}$	$\leftrightsquigarrow$	$n!_{2\mathbb{T}} = (2n)!$	$\leftrightsquigarrow$	$cos^m(\sqrt{x})$
$\mathbb{Z}^2$	$\leftrightsquigarrow$	$n!_{\mathbb{Z}^2} = \frac{(2n)!}{2}$	$\leftrightsquigarrow$	$2cos^m(\sqrt{x})$
$aq^{\mathbb{N}}$	$\leftrightsquigarrow$	$n!_{aq^{\mathbb{N}}} = (-aq)^n q^{\frac{-n(n+1)}{2}}(q:q)_n$	$\leftrightsquigarrow$	?
$\mathbb{P}$	$\leftrightsquigarrow$	$n!_{\mathbb{P}} = \prod_p p^{(stuff)}$	$\leftrightsquigarrow$	$\left(-\frac{ln(1-x)}{x}\right)^m$

Summary of $!_S$-analogues

$\mathbb{N}$	$\leftrightsquigarrow$	$n!_{\mathbb{N}} = n!$	$\leftrightsquigarrow$	$(e^x)^m$
$a\mathbb{N}+b$	$\leftrightsquigarrow$	$n!_{a\mathbb{N}+b} = a^n n!$	$\leftrightsquigarrow$	$(\frac{a}{a-x})^m$
$2\mathbb{T}$	$\leftrightsquigarrow$	$n!_{2\mathbb{T}} = (2n)!$	$\leftrightsquigarrow$	$cos^m(\sqrt{x})$
$\mathbb{Z}^2$	$\leftrightsquigarrow$	$n!_{\mathbb{Z}^2} = \frac{(2n)!}{2}$	$\leftrightsquigarrow$	$2cos^m(\sqrt{x})$
$aq^{\mathbb{N}}$	$\leftrightsquigarrow$	$n!_{aq^{\mathbb{N}}} = (-aq)^n q^{\frac{-n(n+1)}{2}}(q:q)_n$	$\leftrightsquigarrow$	?
$\mathbb{P}$	$\leftrightsquigarrow$	$n!_{\mathbb{P}} = \prod_p p^{(stuff)}$	$\leftrightsquigarrow$	$\left(-\frac{ln(1-x)}{x}\right)^m$
				$(tan^{-1}(x))^m$

Summary of $!_S$-analogues

$\mathbb{N}$	$\leftrightsquigarrow$	$n!_{\mathbb{N}} = n!$	$\leftrightsquigarrow$	$(e^x)^m$
$a\mathbb{N}+b$	$\leftrightsquigarrow$	$n!_{a\mathbb{N}+b} = a^n n!$	$\leftrightsquigarrow$	$(\frac{a}{a-x})^m$
$2\mathbb{T}$	$\leftrightsquigarrow$	$n!_{2\mathbb{T}} = (2n)!$	$\leftrightsquigarrow$	$cos^m(\sqrt{x})$
$\mathbb{Z}^2$	$\leftrightsquigarrow$	$n!_{\mathbb{Z}^2} = \frac{(2n)!}{2}$	$\leftrightsquigarrow$	$2cos^m(\sqrt{x})$
$aq^{\mathbb{N}}$	$\leftrightsquigarrow$	$n!_{aq^{\mathbb{N}}} = (-aq)^n q^{\frac{-n(n+1)}{2}}(q:q)_n$	$\leftrightsquigarrow$	?
$\mathbb{P}$	$\leftrightsquigarrow$	$n!_{\mathbb{P}} = \prod_p p^{(stuff)}$	$\leftrightsquigarrow$	$\left(-\frac{ln(1-x)}{x}\right)^m$
?	$\leftrightsquigarrow$	$n!_?$	$\leftrightsquigarrow$	$(tan^{-1}(x))^m$

Future Work

Conjecture A

Conjecture B

Future Work

Conjecture A

Every subset of $\mathbb{N}$ corresponds to a function.

Conjecture B

Every analytic function corresponds to a subset of $\mathbb{N}$.

Future Work

Conjecture A

Every subset of $\mathbb{N}$ corresponds to a function.

Issues:

Conjecture B

Every analytic function corresponds to a subset of $\mathbb{N}$.

Future Work

Conjecture A

Every subset of $\mathbb{N}$ corresponds to a function.

Issues:

- They probably won't be classical functions.

Conjecture B

Every analytic function corresponds to a subset of $\mathbb{N}$.

Future Work

Conjecture A

Every subset of $\mathbb{N}$ corresponds to a function.

Issues:

- They probably won't be classical functions.
- Difficulty: Finding the correct polynomials in the numerator.

Conjecture B

Every analytic function corresponds to a subset of $\mathbb{N}$.

Future Work

Conjecture A

Every subset of $\mathbb{N}$ corresponds to a function.

Issues:

- They probably won't be classical functions.
- Difficulty: Finding the correct polynomials in the numerator.
- Is it theoretically useful to allow for constants such as for $\mathbb{Z}^2$?

Conjecture B

Every analytic function corresponds to a subset of $\mathbb{N}$.

Future Work

Conjecture A

Every subset of $\mathbb{N}$ corresponds to a function.

Issues:

- They probably won't be classical functions.
- Difficulty: Finding the correct polynomials in the numerator.
- Is it theoretically useful to allow for constants such as for $\mathbb{Z}^2$?

Conjecture B

Every analytic function corresponds to a subset of $\mathbb{N}$.

Issues:

Future Work

Conjecture A

Every subset of $\mathbb{N}$ corresponds to a function.

Issues:

- They probably won't be classical functions.
- Difficulty: Finding the correct polynomials in the numerator.
- Is it theoretically useful to allow for constants such as for $\mathbb{Z}^2$?

Conjecture B

Every analytic function (with conditions?) corresponds to a subset of $\mathbb{N}$.

Issues:

Future Work

Conjecture A

Every subset of $\mathbb{N}$ corresponds to a function.

Issues:

- They probably won't be classical functions.
- Difficulty: Finding the correct polynomials in the numerator.
- Is it theoretically useful to allow for constants such as for $\mathbb{Z}^2$?

Conjecture B

Every analytic function (with conditions?) corresponds to a subset of $\mathbb{N}$.

Issues:

- What are the conditions?

If The Conjectures Are True...

$$
\begin{array}{ccc}
-a\ln(a-x) & \longrightarrow & ? \\
\uparrow {\scriptstyle \int dx} & & \\
\dfrac{a}{a-x} & \longleftrightarrow & a\mathbb{N}+b \\
\downarrow {\scriptstyle \frac{d}{dx}} & & \\
\dfrac{a}{(a-x)^2} & \longrightarrow & ?
\end{array}
$$

Thanks

- References

Bhargava, M. (2000). The factorial function and generalizations. The American Mathematical Monthly, 107(9), 783-799.

Chabert, J.L. (2007). Integer-valued polynomials on prime numbers and logarithm power expansion. European Journal of Combinatorics, 28(3), 754-761.

Thanks

- References

Bhargava, M. (2000). The factorial function and generalizations. The American Mathematical Monthly, 107(9), 783-799.

Chabert, J.L. (2007). Integer-valued polynomials on prime numbers and logarithm power expansion. European Journal of Combinatorics, 28(3), 754-761.

- Thanks Professor Eric Errthum!

Thanks

- References

Bhargava, M. (2000). The factorial function and generalizations. The American Mathematical Monthly, 107(9), 783-799.

Chabert, J.L. (2007). Integer-valued polynomials on prime numbers and logarithm power expansion. European Journal of Combinatorics, 28(3), 754-761.

- Thanks Professor Eric Errthum!
- **Questions**