p-Egyptian Fractions

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Definition and Examples

Definition (Egyptian Fraction)

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- $\frac{5}{121} = \frac{1}{25}$

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- $\frac{5}{121} = \frac{1}{45} + \frac{1}{75} + \frac{1}{300} + \frac{1}{1023} + \frac{1}{1089} + \frac{1}{1860}$
- $\frac{5}{121} = \frac{1}{33} + \frac{1}{121} + \frac{1}{363}$

Basic Facts

Fact (Not Unique)

There is more than one way to expand a rational number into an Egyptian fraction.

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There is more than one way to expand a rational number into an Egyptian fraction.

Fact (Existence)

For all positive rational numbers less than one there exists an Egyptian fraction expansion.

Greedy Method

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$$13 = 4q - r$$

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The **division algorithm** gives the largest q' such that r' remains positive.

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r smaller than 4

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Increment q' by 1 to get the smallest q such that r is positive.

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r smaller than 4 Then $q = 4$ is as small as it can be and produces the largest possible unit fraction $1/4$ that is less than $4/13$.

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$$\frac{4}{13} = \frac{1}{4} + \frac{3}{52} \longleftarrow \text{Not all unit fractions!}$$

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$$\frac{4}{13} = \frac{1}{4} + \frac{1}{18} + \frac{1}{468}$$

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Repeat on $\frac{r_1}{bq_1}$

$$\frac{r_1}{bq_1} - \frac{1}{q_2} = \frac{r_2}{bq_1 q_2}$$

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$$\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_n} = \frac{a}{b}$$

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$$|\frac{30}{189}|_3 = |\frac{2 \cdot 3 \cdot 5}{7 \cdot 3^3}|_3 = |\frac{3}{3^3}|_3 = |\frac{1}{3^2}|_3 = 9 \text{ LARGE}$$

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$|\frac{a}{b}|_p$ a and b contain the same power of p, $|\frac{a}{b}|_p = 1$ SAME

p-Adic Division Algorithm

Definition (Errthum, Lager, 2009)

Let p be an odd prime. Then given any b and $a \in \mathbb{Q}_p$ where $a \neq 0$, there exists uniquely $q' = \frac{m}{p^k}$ with $|q'|_< p$, and $r' \in \mathbb{Q}_p$ with $|r'|_p < |a|_p$ such that $b = aq' + r'$.

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$\frac{5}{6}$ divided by $-\frac{2}{3}$ is 4 with a remainder of $\frac{7}{2}$ in the 7-adics.

$$\frac{5}{6} = \left(-\frac{2}{3}\right) 4 + \frac{7}{2}$$

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20 divided by 5 is 4 with a remainder of 0 in the 7-adics.

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20 divided by 5 is 1 with a remainder of 15 in the 3-adics.

$$\frac{5}{6} = \left(-\frac{2}{3}\right)4 + \frac{7}{2} \qquad 20 = 5 \cdot 4 + 0 \qquad 20 = 5 \cdot 1 + 15$$

p-Adic Greedy Algorithm

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Take $q = q' + ?$ so that $b = aq - r$.

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In the classical case,

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In the p-adic case,

$$q = q' + \left\lceil \frac{\frac{b}{a} - q'}{p} \right\rceil p$$

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In the p-adic case,

$$q = q' + \left\lceil \frac{\frac{b}{a} - q'}{p} \right\rceil p$$

Choose $p = 1$ and pretend it's the classical case,

$$q = q' + \overbrace{\left\lceil \frac{\frac{b}{a} - \lfloor \frac{b}{a} \rfloor}{1} \right\rceil}^{\text{Just equal to 1}} 1$$

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Take $q = q' + \left\lceil \frac{\frac{b}{a} - q'}{p} \right\rceil p$ so that
$b = aq - r$.

Different kind of Egyptian...

p-Adic Egyptian Fraction

Instead of fractions of the form,

$$\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n}$$

with distinct terms, allow for increasing whole powers of a prime p to appear in the numerator,

$$\frac{p^{\varepsilon_1}}{d_1} + \frac{p^{\varepsilon_2}}{d_2} + \frac{p^{\varepsilon_3}}{d_3} + \cdots + \frac{p^{\varepsilon_n}}{d_n}$$

with $0 \leq \varepsilon_n < \varepsilon_{n+1}$.

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$$9 = 2 \cdot 2 + 5$$

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$$= 2 + \lceil \frac{1}{2} \rceil 5 = 7$$

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$$9 \cdot 7 = 5 \left(\frac{13}{5} \right) + 50$$

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$$= \frac{13}{5} + \lceil 2 \rceil 5 = \frac{63}{5} \qquad 63 = 5 \left(\frac{63}{5} \right) - 0$$

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$$9 \cdot 7 = 5\left(\frac{13}{5}\right) + 50 \qquad q_2 = \frac{13}{5} + \lceil \frac{\frac{63}{5} - \frac{13}{5}}{5} \rceil 5$$

$$= \frac{13}{5} + \lceil 2 \rceil 5 = \frac{63}{5} \qquad 63 = 5\left(\frac{63}{5}\right) - 0$$

$$\frac{1}{7} + \frac{5}{63} = \frac{2}{9}$$

Alternate Egyptian fraction expansion

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$$\frac{a}{b \cdot p^{\varepsilon_n}} = \frac{1}{d_1 \cdot p^{\varepsilon_n - \varepsilon_1}} + \frac{1}{d_2 \cdot p^{\varepsilon_n - \varepsilon_2}} + \cdots + \frac{1}{d_n}$$

Alternate Egyptian fraction expansion

$$\frac{a}{b} = \frac{p^{\varepsilon_1}}{d_1} + \frac{p^{\varepsilon_2}}{d_2} + \cdots + \frac{p^{\varepsilon_n}}{d_n}$$
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How do we make $\frac{a}{b \cdot p^{\varepsilon_n}}$ the input?

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