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Dedicated to Professor
Stefan Straszewicz

Jerzy Browkin

Continued fractions in local fields, I

Dedicated to Professor Stefan Straszewicz

0. Introduction

In the present paper we define a continued fraction expansion of an element of a field complete with respect to a discrete valuation and we prove some basic properties of such continued fractions. There is some analogy with the ordinary continued fraction expansions of real numbers.

A.A.Ruban [3] considered analogous continued fraction expansions of p-adic integers and stated the existence and the uniqueness of such expansions. There are also considered another continued fractions connected with p-adic numbers (see e.g. [1] and [5]). It seems however that these continued fractions have very few in common with our definition.

We shall use the standard notations of the book of Perron [2].

1. The mapping s. Let K be a field complete with respect to a discrete valuation v satisfying $v(K^*) = Z$. Denote by ϑ the canonical homomorphism of additive groups $K \to K/m_v$, where m_v is the maximal ideal of the ring O_v of integers of K.

We shall consider a mapping $s: K \to K$ satisfying $s(0) = 0$, $\vartheta s = \vartheta$ and $s(a) = s(b)$ for $a-b \in m_v$. It follows that $s(b) - b \in m_v$ for every $b \in K$. Of course such a mapping s is not canonical. In some important particular cases we fix the mapping s as follows.

(1) Every element a of the field K complete with respect to a discrete valuation v can be uniquely written in the form

$$a = \sum_{n=r}^{\infty} a_n \pi^n, \tag{1.1}$$

where $v(\pi) = 1$, $r \in Z$ and coefficients a_n belong to a fixed set R of representatives of the residue class field $\bar{K}$ of K. We assume that $0 \in R$ and $a_r \neq 0$. For such an element a we define

$$s(a) = \sum_{n=r}^{0} a_n \pi^n. \tag{1.2}$$

It is easy to verify that the mapping s defined by (1.1) and (1.2) has the required properties.

In particular if $K = k((x))$ is the field of power series over a field k then we define the mapping s as follows

$$s\left(\sum_{n=r}^{\infty} a_n x^n\right) = \sum_{n=r}^{0} a_n x^n, \tag{1.3}$$

where $a_n \in k$ and $r \in Z$. From this formula we observe that in the case $K = k((x))$ the set s(K) is a ring, namely the ring of polynomials $k[x^{-1}]$.

(2) Let $K = Q_p$ be the field of p-adic numbers and denote by v the p-adic valuation. Let us observe that every coset of Q_p/m_v has the unique representative belonging to $Z\left[\frac{1}{p}\right]$ and to the interval $\left(-\frac{p}{2}, \frac{p}{2}\right>$. Denote this representative of the coset $a + m_v$ by s(a). It is easy to verify that the mapping s has the required properties.

If we represent a p-adic number a (where p is an odd prime) as the series

$$a = \sum_{n=r}^{\infty} a_n\, p^n,$$

where $r \in Z$ and $a_n \in \left\{0, \pm 1, \pm 2, \ldots, \pm \frac{1}{2}(p-1)\right\}$, then

$$\left|\sum_{n=r}^{0} a_n\, p^n\right| \leqslant \frac{1}{2}(p-1)\sum_{n=r}^{0} p^n = \frac{1}{2}\,\frac{p^{r+1}-1}{p^r} < \frac{p}{2}$$

and evidently

$$\sum_{n=r}^{0} a_n\, p^n \in Z\left[\frac{1}{p}\right]. \quad \text{Therefore} \quad s(a) = \sum_{n=r}^{0} a_n\, p^n.$$

It follows that in the case $K = Q_p$, p odd and $R = \left\{0, \pm 1, \pm 2, \ldots, \pm \frac{1}{2}(p-1)\right\}$ mappings s given by (1) and (2) coincide. For $p = 2$ these mappings are different.

In what follows we assume that the mapping s is fixed.

2. Continued fractions. For any element $\xi_o \in K$ we define sequences (ξ_n) and (b_n) as follows

$$(2.1) \quad \xi_{n+1} = (\xi_n - s(\xi_n))^{-1} \quad \text{if} \quad \xi_n \neq s(\xi_n), \quad b_n = s(\xi_n).$$

If $\xi_n = s(\xi_n)$ for some n then the element ξ_{n+1} is not defined and the sequences (ξ_n) and (b_n) are finite.

We have

$$\xi_o = b_o + \frac{1|}{|b_1} + \frac{1|}{|b_2} + \ldots + \frac{1|}{|b_n} + \frac{1|}{|\xi_{n+1}} = [b_o; b_1, b_2, \ldots, b_n, \xi_{n+1}].$$

Let us observe that $\xi_n - s(\xi_n) \in m_v$ i.e. $v(\xi_n - s(\xi_n)) > 0$ Hence $v(\xi_{n+1}) = -v(\xi_n - s(\xi_n)) < 0$ for $n \geqslant 0$. Consequently $v(b_{n+1}) = v(s(\xi_{n+1})) = v(\xi_{n+1} + (s(\xi_{n+1}) - \xi_{n+1})) = v(\xi_{n+1}) < 0$ for $n \geqslant 0$. Moreover

$$v(b_o) = v(s(\xi_o)) = \begin{cases} v(\xi_o) & \text{if} \quad v(\xi_o) \leqslant 0 \\ v(0) & \text{if} \quad v(\xi_o) > 0 \end{cases}$$

i.e. $b_o = 0$ or $v(b_o) = v(\xi_o) \leqslant 0$.

For any elements $b_o, b_1, \ldots$ of K let $A_n = A_n(b_o, b_1, \ldots, b_n)$ and $B_n = B_n(b_1, b_2, \ldots, b_n)$ be defined as follows:

$$\text{(2.2)} \quad \begin{aligned} A_o &= b_o, \quad A_1 = b_o b_1 + 1, \quad A_{n+2} = b_{n+2} A_{n+1} + A_n, \\ B_o &= 1, \quad B_1 = b_1, \quad B_{n+2} = b_{n+2} B_{n+1} + B_n. \end{aligned}$$

L e m m a 1. If elements $b_1, b_2, \ldots \in K$ satisfy $v(b_n) < 0$ for $n \geqslant 1$ and B_n are defined by (2.2) then

$$v(B_n) = v(b_1) + v(b_2) + \ldots + v(b_n) \quad \text{for} \quad n \geqslant 0.$$

P r o o f . We have $v(B_o) = v(1) = 0$ and $v(B_1) = v(b_1)$, i.e. the lemma holds for $n = 0$ and $n = 1$. Let us assume that $v(B_n) = v(b_1) + v(b_2) + \ldots + v(b_n)$ and $v(B_{n+1}) = v(b_1) + v(b_2) + \ldots + v(b_n) + v(b_{n+1})$ for some non negative integer n. Then

$$v(b_{n+2} B_{n+1}) = v(b_{n+2}) + v(b_1) + v(b_2) + \ldots + v(b_{n+1}) < v(B$$

Therefore

$$v(B_{n+2}) = v(b_{n+2} B_{n+1} + B_n) = v(b_{n+2} B_{n+1}) = v(b_{n+2}) + v(b_1)$$

$$+ v(b_2) + \ldots + v(b_n) + v(b_{n+1})$$

and the lemma follows by induction.

C o r o l l a r y . Under the assumptions of the lemma we have $v(B_n) < 0$ for $n \geqslant 1$ and hence $B_n \neq 0$ for $n \geqslant 0$.

From (2.2) it follows that A_n and B_n are respectively the numerator and the denominator of the continued fraction $\left[b_o; b_1, b_2, \ldots, b_n\right]$ i.e. $\left[b_o; b_1, b_2, \ldots, b_n\right] = \frac{A_n}{B_n}$.

Theorem 1. If elements $b_o, b_1, \ldots \in K$ satisfy $v(b_n) < 0$ for $n \geqslant 1$ and A_n, B_n are defined by (2.2) then the sequence $\left(\frac{A_n}{B_n}\right)$ is convergent to an element $g \in K$ and

$$v\left(g - \frac{A_n}{B_n}\right) = v(B_nB_{n+1}) \quad \text{for} \quad n \geqslant 0.$$

In particular $v(g - b_o) > 0$.

Proof. By the well known formulas we have

$$\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = \frac{A_{n+1}B_n - A_nB_{n+1}}{B_nB_{n+1}} = \frac{(-1)^n}{B_nB_{n+1}} .$$

Hence

$$v\left(\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n}\right) = - v(B_nB_{n+1}) > 0 \quad \text{for} \quad n \geqslant 0.$$

Therefore $\left(\frac{A_n}{B_n}\right)$ is a Cauchy sequence and it is convergent in the complete field K.

Since by Lemma 1 the sequence $(-v(B_nB_{n+1}))$ is increasing we have

$$v\left(\frac{A_m}{B_m} - \frac{A_n}{B_n}\right) = -v(B_nB_{n+1}) \quad \text{for every} \quad m > n.$$

If $m \to \infty$ we obtain $v\left(g - \frac{A_n}{B_n}\right) = -v(B_nB_{n+1})$, as claimed. Since $\frac{A_o}{B_o} = b_o$ we have in particular $v(g - b_o) = -v(B_oB_1) = -v(b_1) > 0$.

We denote $\lim_{n\to\infty} \frac{A_n}{B_n}$ by $\left[b_o; b_1, b_2, \ldots\right]$.

Assume now that the sequences (b_n), (b'_n) satisfy $b_n, b'_n \in s(K)$ for $n \geq 0$ and $v(b_n) < 0$, $v(b'_n) < 0$ for $n \geq 1$. By standard arguments it follows that if these sequences are different then the sequences $[b'_0; b'_1, b'_2, \ldots, b'_n]$ and $[b_0; b_1, b_2, \ldots, b_n]$ have different limits.

Namely we can assume that $b_0 \neq b'_0$ and then by Theorem we have $v(g-b_0) > 0$ and $v(g'-b'_0) > 0$, where $g = \lim_{n\to\infty} [b_0; b_1, \ldots, b_n]$ $g' = \lim_{n\to\infty} [b'_0; b'_1, \ldots, b'_n]$. The equality $g = g'$ would imply that b_0 and b'_0 belong to the same coset modulo m_v and hence $b_0 = b'_0$. Contradiction.

T h e o r e m 2. Assume that the sequences (ξ_n) and (b_n) defined for an element $\xi_0 \in K$ by (2.1) are infinite and define A_n and B_n by (2.2). Then the sequence $\left(\frac{A_n}{B_n}\right)$ is convergent to ξ_0.

P r o o f . We have

$$\xi_0 = \frac{A_{n+1}(b_0, b_1, \ldots, b_n, \xi_{n+1})}{B_{n+1}(b_1, b_2, \ldots, b_n, \xi_{n+1})} = \frac{\xi_{n+1}A_n + A_{n-1}}{\xi_{n+1}B_n + B_{n-1}} .$$

Therefore

$$\xi_0 - \frac{A_n}{B_n} = \frac{A_{n-1}B_n - A_nB_{n-1}}{(\xi_{n+1}B_n + B_{n-1})B_n} = \frac{(-1)^n}{(\xi_{n+1}B_n + B_{n-1})B_n} .$$

We have $v(\xi_{n+1}) = v(b_{n+1}) < 0$. Hence from Lemma 1 it follows that $v(\xi_{n+1}B_n + B_{n-1}) = v(B_{n+1})$ and consequently

$$\lim_{n\to\infty}\left(\xi_0 - \frac{A_n}{B_n}\right) = 0.$$

From Theorem 2 it is easy to deduce that every element $\xi_0 \in K$ has the unique finite or infinite continued fraction expansion $\xi_0 = [b_0; b_1, b_2, \ldots]$, where $b_n \in s(K)$ for $n \geq 0$ and $v(b_n) < 0$ for $n \geq 1$.

In the sequel we shall consider only the continuous fraction expansions of this form.

3. Continued fractions of rational numbers. In the field Q_p of p-adic numbers we consider the mapping s defined by (2), i.e. $s(a) \in Z\left[\frac{1}{p}\right]$, $-\frac{p}{2} < s(a) \leqslant \frac{p}{2}$ and $v(a - s(a)) > 0$ for any $a \in K$.

T h e o r e m 3. Every rational number has a finite continued fraction expansion in the field Q_p for every p.

P r o o f . Let $\xi_o \in Q \subset Q_p$ and define sequences (ξ_n) and (b_n) by means of (2.1) using the p-adic valuation v. We have for $n \geqslant 1$

$$\xi_n = b_n + \xi_{n+1}^{-1} , \tag{3.1}$$

if ξ_{n+1} is defined. Here $b_n = s(\xi_n)$ is a rational number belonging to $Z\left[\frac{1}{p}\right]$ and to the interval $\left(-\frac{p}{2} , \frac{p}{2}\right>$. Therefore $b_n = c_n \cdot p^{-k}$, where $|c_n| \leqslant \frac{1}{2} p^{k+1}$, $c_n \in Z$ and $-k = v(b_n) = v(\xi_n) < 0$. It follows that

$$\xi_n = \frac{\alpha_n}{p^k \beta_n} , \tag{3.2}$$

where $\alpha_n, \beta_n \in Z$, $(\alpha_n, \beta_n) = 1$ and $p \nmid \alpha_n \beta_n$.

Analogously we prove that

$$\xi_{n+1} = \frac{\alpha_{n+1}}{p^m \beta_{n+1}} , \tag{3.3}$$

where $m \geqslant 1$, $\alpha_{n+1}, \beta_{n+1} \in Z$, $(\alpha_{n+1}, \beta_{n+1}) = 1$ and $p \nmid \alpha_{n+1} \beta_{n+1}$.

From (3.1) and (3.2) it follows that $\xi_{n+1} = (\xi_n - b_n)^{-1} = p^k \beta_n (\alpha_n - c_n \beta_n)^{-1}$ and hence by (3.3) we obtain

$$\alpha_{n+1}(\alpha_n - c_n \beta_n) = p^{k+m} \beta_n \beta_{n+1} . \tag{3.4}$$

From (3.4) we observe that $\alpha_{n+1} = \pm \beta_n$ and $\beta_{n+1} = \pm p^{-k-m} \cdot (\alpha_n - c_n \beta_n)$. It follows that

$$|\beta_{n+1}| \leqslant p^{-k-m}\left(|\alpha_n| + \tfrac{1}{2}\, p^{k+1}\, |\beta_n|\right) < \tfrac{1}{2}\,|\alpha_n| + \tfrac{1}{2}\,|\beta_n| \,.$$

Therefore

$$|\alpha_{n+1}| + 2\,|\beta_{n+1}| < |\beta_n| + \left(|\alpha_n| + |\beta_n|\right) = |\alpha_n| + 2\,|\beta_n| .$$

Since the sequence $(|\alpha_n| + 2|\beta_n|)$ consists of natural numbers and decreases, then it is finite.

4. Periodic continued fractions. Let K be a field complete with respect to a discrete valuation v and let S be the field generated by the set $s(K)$. If for $\xi_0 \in K$ we have a periodic continued fraction expansion

$$\xi_0 = \left[b_0;\ b_1,\ b_2,\ldots,b_r,\ \overline{b_{r+1},\ldots,b_t}\right],$$

with the period $(b_{r+1},\ldots,b_t)$, where $b_i \in s(K)$ for $0 \leqslant i \leqslant t$, $v(b_j) < 0$ for $j \geqslant 1$, then by the standard arguments it follows that ξ_0 belongs to a quadratic extension of the field S.

The converse is not true in general but in several particular cases we can give some positive results. On the other hand we are unable to decide if every $\xi_0 \in Q_p$ such that $(Q(\xi_0) : Q) = 2$ has a periodic continued fraction expansion in Q_p.

We begin with a description of the general procedure. Let $D \in S$, $\sqrt{D} \in K$ and $(S(\sqrt{D}) : S) = 2$. Moreover suppose that $v(2) = 0$. Consider sequences (ξ_n) and (b_n) defined by means of (2.1) for the element $\xi_0 = \sqrt{D}$. It follows that $b_n \in S$ and $\xi_n \in S(\sqrt{D})$. Therefore there exist uniquely determined elements $P_n, Q_n \in S$ such that

$$\xi_n = \frac{\sqrt{D} + P_n}{Q_n} \,.$$

The sequences (P_n) and (Q_n) can be described as follows:

$$(4.1)\quad \begin{aligned} &P_o = 0, \quad Q_o = 1, \quad P_1 = b_o, \quad Q_1 = D - b_o^2, \\ &P_{n+1} = b_nQ_n - P_n, \; Q_{n+1} = b_n(P_n - P_{n+1}) + Q_{n-1} \text{ for } n \geqslant 1. \end{aligned}$$

It is easy to deduce that

$$(4.2)\qquad D - P_{n+1}^2 = Q_nQ_{n+1} \quad \text{for} \quad n \geqslant 0.$$

Let $\sigma \in G(S(\sqrt{D})/S)$ satisfy $\sigma(\sqrt{D}) = -\sqrt{D}$. We define $\eta_n = \sigma(\xi_n)$. It follows that

$$\eta_n = \frac{-\sqrt{D} + P_n}{Q_n}.$$

Applying the automorphism σ to the equality $\xi_{n+1} = (\xi_n - b_n)^{-1}$ we obtain $\eta_{n+1} = (\eta_n - b_n)^{-1}$, because $b_n \in S$.

L e m m a 2. $v(\eta_{n+1}) = -v(\xi_n)$ for $n \geqslant 0$.

P r o o f . Suppose that $v(\sqrt{D}) \leqslant 0$. We have $\eta_1 = (\eta_o - b_o)^{-1} = (\eta_o - \xi_o + \xi_1^{-1})^{-1} = (-2\,\xi_o + \xi_1^{-1})^{-1}$, because $\xi_o - b_o = \xi_1^{-1}$ and $\eta_o = -\sqrt{D} = -\xi_o$. Since $v(\xi_1^{-1}) > 0$ and $v(\xi_o) = v(\sqrt{D}) \leqslant 0$, we obtain $v(\eta_1) = -v(-2\,\xi_o + \xi_1^{-1}) = -v(\xi_o)$. Similarly $\eta_2 = (\eta_1 - b_1)^{-1} = (\eta_1 - \xi_1 + \xi_2^{-1})^{-1}$ and hence $v(\eta_2) = -v(\eta_1 - \xi_1 + \xi_2^{-1}) = -v(\xi_1)$, because $v(\xi_2^{-1}) > 0$, $v(\eta_1) = -v(\xi_o) = -v(\sqrt{D}) \geqslant 0$ and $v(\xi_1) < 0$.

Now let $v(\sqrt{D}) > 0$. We have $b_o = s(\sqrt{D}) = 0$ and it follows that

$$\eta_1 = (\eta_o - b_o)^{-1} = \eta_o^{-1} = (-\sqrt{D})^{-1} = -\xi_o^{-1}.$$

Hence $v(\eta_1) = -v(\xi_o)$. Analogously we have $\xi_1 = (\xi_o - b_o)^{-1} = \xi_o^{-1} = (\sqrt{D})^{-1}$ and hence $\eta_1 = -(\sqrt{D})^{-1} = -\xi_1$. Moreover $\xi_1 - b_1 = \xi_2^{-1}$, where $v(\xi_2^{-1}) > 0$. Consequently

$$\eta_2 = (\eta_1 - b_1)^{-1} = (\eta_1 - \xi_1 + \xi_2^{-1})^{-1} = (-2\xi_1 + \xi_2^{-1})^{-1}.$$

Hence $v(\eta_2) = -v(-2\xi_1 + \xi_2^{-1}) = -v(\xi_1)$.

So the lemma holds for $n = 1$ and $n = 2$. Now we proceed by induction. For $n \geqslant 2$ we have $\eta_{n+1} = (\eta_n - b_n)^{-1}$ and $v(\eta_n) = -v(\xi_{n-1}) > 0$ by the inductive assumption. Since $v(b_n) = v(\xi_n) < 0$, it follows that $v(\eta_{n+1}) = -v(\eta_n - b_n) = -v(b_n) = -v(\xi_n)$.

L e m m a 3. For $n \geqslant 2$ we have

(i) $$v(P_n) = v(Q_n b_n) = v(\sqrt{D}),$$

(ii) $$v(P_n - \sqrt{D}) = v(\sqrt{D}) - v(b_n b_{n+1}).$$

P r o o f . We have $v(\xi_n) < 0$ and $v(\eta_n) = -v(\xi_{n-1}) > 0$ for $n > 2$. Hence $v(\xi_n \pm \eta_n) = v(\xi_n)$, i.e.

$$v\left(\frac{2P_n}{Q_n}\right) = v(b_n) \quad \text{and} \quad v\left(\frac{2\sqrt{D}}{Q_n}\right) = v(b_n).$$

It proves the first part of the lemma. The second part can be deduced from the definition of η_n as follows

$$v\left(\frac{P_n - \sqrt{D}}{Q_n}\right) = v(\eta_n) = -v(\xi_{n-1}) = -v(b_{n-1}).$$

Consequently

$$v(P_n - \sqrt{D}) = v(Q_n) - v(b_{n-1}) = v(b_n Q_n) - v(b_n b_{n-1}) =$$

$$= v(\sqrt{D}) - v(b_n b_{n-1}).$$

T h e o r e m 4. If $K = k((x))$ is the field of power series over a finite field k, then every element $\sqrt{D} \in K \setminus S$ such that $D \in s(K)$ has the periodic continued fraction expansion.

P r o o f . For the field $K = k((x))$ of power series over any field k we have $s(K) = k[x^{-1}]$ and $S = k(x)$. Define elements P_n and Q_n by means of (4.1). Since the set $s(K)$ is a ring and $D \in s(K)$, then from the formulas (4.1) it follows that $P_n, Q_n \in s(K)$, i.e. P_n and Q_n are polynomials in x^{-1}.

From Lemma 3 it follows that the polynomials P_n have the same degree and degrees of the polynomials Q_n are bounded. Consequently if the field k is finite, then the sequence (P_n, Q_n), $n = 0, 1, 2, \ldots$, has only finite number of distinct terms. Since this sequence is defined by recurrent relations, it is periodic. It follows that the sequence (b_n) is also periodic.

On the other hand if the field k is infinite, then Theorem 4 does not hold in general (see the paper [2] of Schinzel).

Let $D \in S$ and $\sqrt{D} \in K \setminus S$. We shall consider the equation

$$x^2 - Dy^2 = (-1)^n. \tag{4.3}$$

A solution $x = t$, $y = u \neq 0$ of the equation (4.3) is said to be a standard solution, if

(i) The element $\frac{t}{u}$ has a finite continued fraction expansion in K, $\frac{t}{u} = [b_0; b_1, b_2, \ldots, b_r]$,

(ii) $t = A_r(b_0, b_1, \ldots, b_r)$, $u = B_r(b_1, b_2, \ldots, b_r)$,

(iii) $r \equiv n-1 \pmod 2$.

Let us remark that in the case $K = Q_p$ we have $S = Q$ and hence by Theorem 3 the condition (i) is always fulfilled.

The following theorem gives some connections between the periodicity of the continued fraction expansion of the element $\sqrt{D}$ and the existence of a standard solution of the equation (4.3).

T h e o r e m 5. Let $D \in S$ and $\sqrt{D} \in K \setminus S$. Then

$$\sqrt{D} = [b_0; \overline{b_1, b_2, \ldots, b_q}] \quad \text{in} \quad K \tag{4.4}$$

if and only if $x = A_{q-1}(b_o,b_1,\dots,b_{q-1})$, $y = B_{q-1}(b_1,b_2,\dots$ $\dots,b_{q-1})$ is a standard solution of the equation (4.3) and

$$b_q = b_o + \frac{A_{q-1} - B_{q-2}}{B_{q-1}} \tag{4.5}$$

belongs to $s(K)$.

P r o o f . $\Longrightarrow$. Suppose that (4.4) holds. Then

$$\sqrt{D} = \left[b_o;\ b_1,\dots,b_{q-1},\ b_q - b_o + \sqrt{D}\right].$$

It follows that

$$\sqrt{D} = \frac{A_{q-1}(b_q - b_o \sqrt{D}) + A_{q-2}}{B_{q-1}(b_q - b_o \sqrt{D}) + B_{q-2}} \tag{4.6}$$

and henc

$$\sqrt{D}\ (B_{q-1}(b_q - b_o) + B_{q-2}) + DB_{q-1} = \sqrt{D}\ A_{q-1} + A_{q-1}(b_q - b_o) + A_{q-2}$$

From the irrationality of $\sqrt{D}$ over S we deduce that

$$\begin{aligned} B_{q-1}(b_q - b_o) + B_{q-2} &= A_{q-1}, \\ A_{q-1}(b_q - b_o) + A_{q-2} &= DB_{q-1}. \end{aligned} \tag{4.7}$$

Multiplying the first of the above equalities by A_{q-1}, the second - by $-B_{q-1}$ and adding the results we obtain

$$A_{q-1}^2 - D\ B_{q-1}^2 = A_{q-1}B_{q-2} - A_{q-2}B_{q-1} = (-1)^{q-2} = (-1)^q.$$

Therefore (A_{q-1}, B_{q-1}) is a standard solution of the equation $t^2 - D\,u^2 = (-1)^q$. Moreover from (4.7) it follows that

$$b_o + \frac{A_{q-1} - B_{q-2}}{B_{q-1}} = b_q$$

and hence this element belongs to $s(K)$.

$\Leftarrow$. Let t, u be a standard solution of the equation (4.3). It means that $u \neq 0$ and $\frac{t}{u}$ has a finite continued fraction expansion $\frac{t}{u} = [b_o; b_1, b_2, \dots, b_{q-1}]$ such that $t = A_{q-1}$, $u = B_{q-1}$ and $q-1 \equiv n \pmod 2$. Moreover let

$$b_q = b_o + \frac{A_{q-1} - B_{q-2}}{B_{q-1}} \in s(K). \tag{4.8}$$

Since $A_{q-1}^2 - D\,B_{q-1}^2 = (-1)^q = B_{q-2}A_{q-1} - A_{q-2}B_{q-1}$, then from (4.8) we deduce that

$$b_q - b_o = \frac{A_{q-1} - B_{q-2}}{B_{q-1}} = \frac{D\,B_{q-1} - A_{q-2}}{A_{q-1}}. \tag{4.9}$$

We shall prove that (4.6) holds. In fact, from (4.9) it follows that

$$\frac{A_{q-1}(b_q - b_o + \sqrt{D}) + A_{q-2}}{B_{q-1}(b_q - b_o + \sqrt{D}) + B_{q-2}} = \frac{A_{q-1}\sqrt{D} + (D\,B_{q-1} - A_{q-2}) + A_{q-2}}{B_{q-1}\sqrt{D} + (A_{q-1} - B_{q-2}) + B_{q-2}} =$$

$$= \frac{D\,B_{q-1} + \sqrt{D}\,A_{q-1}}{\sqrt{D}\,B_{q-1} + A_{q-1}} = \sqrt{D}.$$

From (4.6) we deduce that

$$\sqrt{D} = [b_o; b_1, b_2, \dots, b_{q-1}, b_q - b_o + \sqrt{D}] \tag{4.10}$$

and hence

$$\sqrt{D} - b_o = [0; b_1, b_2, \dots, b_{q-1}, b_q + (\sqrt{D} - b_o)]. \tag{4.11}$$

Therefore substituting (4.11) into (4.10) and applying standard arguments we obtain that $\sqrt{D} = [b_o; \overline{b_1, b_2, \dots, b_{q-1}, b_q}]$.

5. Examples. We cannot decide if for every $D \in Q$ such that $(Q(\sqrt{D}) : Q) = 2$ and $\sqrt{D} \in Q_p$ the continued fraction for $\sqrt{D}$ in Q_p is periodic. We give below some numerical examples in the case p 5. In all these examples the continued fraction of $\sqrt{D}$ in Q_5 is periodic with an even period.

To find a period of the sequence (b_n) it is sufficient to find k and m such that $P_k = P_{k+m}$ and $Q_k = Q_{k+m}$.

It follows then that $\xi_k = \dfrac{\sqrt{D} + P_k}{Q_k} = \xi_{k+m}$ and consequently $b_{k+t} = b_{k+m+t}$ for all $t \geqslant 0$. Of course D should satisfy $2 \mid v(D)$, $\sqrt{D} \notin Q$ and $D \cdot 5^{-v(D)} \equiv \pm 1 \pmod 5$.

D = 6. $\sqrt{6} = 1 - 2.5 + 1.5^2 + \dots$

n	0	1	2	3	4	5	6
b_n	1	-8/5	6/5	7/5	-16/25	7/5	6/5
P_n	0	1	-9	-9	16	16	-9
Q_n	1	5	-15	-5	-50	5	-15

$$\sqrt{6} = [1;\ -8/5,\ \overline{6/5,\ 7/5,\ -16/25,\ 7/5}]$$

D = 11. $\sqrt{11} = 1 + 1.5 + 2.5^2 + 0.5^3 + 0.5^4 + \dots$

n	0	1	2	3	4	5	6	7	8	9	10
b_n	1	-9/5	9/5	-8/5	9/5	6/5	2/5	56/25	-2/5	56/25	2/5
P_n	0	1	-19	-44	-44	-19	31	-69	181	181	-69
Q_n	1	10	-35	55	-35	10	-95	50	-655	50	-95

n	11	12
b_n	6/5	9/5
P_n	31	-19
Q_n	10	-35

$$\sqrt{11} = [1;\ -9/5,\ \overline{9/5,\ -8/5,\ 9/5,\ 6/5,\ 2/5,\ 56/25,\ -2/5,\ 56/25,\ 2/5,\ 6/5}].$$

$\underline{D = 14}$. $\sqrt{14} = 2 - 2.5 + 2.5^2 - 1.5^3 + \dots$.

n	0	1	2	3	4	5	6	7	8
b_n	2	-3/5	-9/5	-6/5	166/125	-6/5	-9/5	-8/5	-9/5
P_n	0	2	-8	17	-83	-83	17	-8	-8
Q_n	1	10	-5	55	-125	55	-5	10	-5

$$\sqrt{14} = [2;\ -3/5,\ \overline{-9/5,\ -6/5,\ 166/125,\ -6/5,\ -9/5,\ -8/5}].$$

$D = \frac{6}{25}$. $\sqrt{D} = \frac{1}{5} - 2 + 1.5 - 1.5^2 - 2.5^3 + \dots$.

n	0	1	2	3	4	5
b_n	-9/5	6/5	7/5	-16/25	7/5	6/5
P_n	0	-9/5	-9/5	16/5	16/5	-9/5
Q_n	1	-3	1	-10	1	-3

$$\frac{\sqrt{6}}{5} = [-9/5;\ \overline{6/5,\ 7/5,\ -16/25,\ 7/5}].$$

$D = \frac{11}{25}$. $\sqrt{D} = \frac{1}{5} + 1 + 2.5 + \dots$

n	0	1	2	3
b_n	6/5	-12/5	12/5	-12/5
P_n	0	6/5	6/5	6/5
Q_n	1	-1	1	-1

$$\frac{\sqrt{11}}{5} = [6/5;\ \overline{-12/5,\ 12/5}].$$

$D = \frac{14}{25}$. $\sqrt{D} = \frac{2}{5} - 2 + 2.5 - 1.5^2 + \dots$

n	0	1	2	3	4	5	6	7
b_n	-8/5	8/5	9/5	6/5	-166/125	6/5	9/5	8/5
P_n	0	-8/5	-8/5	17/5	-83/5	-83/5	17/5	-8/5
Q_n	1	-2	1	-11	25	-11	1	-2

$$\frac{14}{5} = [-8/5;\ \overline{8/5,\ 9/5,\ 6/5,\ -166/125,\ 6/5,\ 9/5}].$$

On the other hand our attempts to prove that the continued fraction of $\sqrt{19}$ in Q_5 is periodic were without success and after determining 22 terms of the sequence (b_n) we did not obtain a period.

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