Supplementary Materials for Chapter 8

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Motivating Identity

Although we can blissfully ignore the connections between complex numbers and trigonometry, it requires a heck of a lot more work and the notation is uninstructive. Instead we use the following identity which cannot be formally proved until Calculus II:

Theorem (Euler's Identity)

For any real number θ,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where i is the imaginary unit, i.e. $i^2 = -1$.

Examples

Example

1. $e^{i\pi} = \cos\pi + i\sin\pi = -1.$
2. $e^{i3\pi/2} = \cos 3\pi/2 + i\sin 3\pi/2 = -i.$
3. $e^{i8\pi} = \cos 8\pi + i\sin 8\pi = 1.$
4. $6e^{i\pi/3} = 6(\cos\pi/3 + i\sin\pi/3) = 6(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 3 + i3\sqrt{3}.$
5. $\sqrt{2}e^{\pi/4} = \sqrt{2}(\cos\pi/4 + i\sin\pi/4) = \sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = 1 + i.$

Complex Numbers in Polar Form

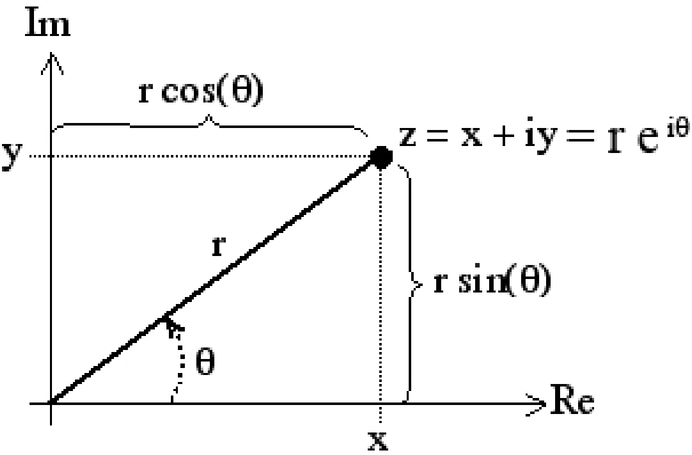
Definition

A complex number $z = a + bi$ has **polar form**:

$$z = re^{i\theta}$$

where $r = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$. The value r is called the modulus of the complex number and is often denoted $r = |z|$. The value θ is called the argument of the complex number and is sometimes denoted $\theta = \arg z$.

The picture on the next slide pulls together our rectangular view of complex numbers and the polar form through Euler's Identity.


Im
r cos(θ)
y
z = x + iy = r e iθ
r
r sin(θ)
θ
Re
x

Example (Example 5 from Section 8.3)

Write each complex number in polar form.

1. $1+i$
2. $-1+\sqrt{3}i$
3. $-4\sqrt{3}-4i$
4. $3+4i$

SOLUTIONS:

1. The argument associated to a positive r is $\theta=\pi/4$. Then $r=\sqrt{1^2+1^2}=\sqrt{2}$. Thus $1+i=\sqrt{2}e^{i\pi/4}$.
2. The argument associated to a positive r is $\theta=2\pi/3$. Then $r=\sqrt{1+3}=2$. Thus $-1+\sqrt{3}i=2e^{i2\pi/3}$.
3. The argument associated to a positive r is $\theta=7\pi/6$. Then $r=\sqrt{48+16}=8$. Thus $-4\sqrt{3}-4i=8e^{i7\pi/6}$.
4. The argument associated to a positive r is $\theta=\tan^{-1}(4/3)\approx 0.927$. Then $r=\sqrt{3^2+4^2}=5$. Thus $3+4i=5e^{i0.927}$.

Multiplication and Division of Complex Numbers in Polar Form

Theorem

If $z_1 = r_1 e^{i\theta_1}$ *and* $z_2 = r_2 e^{i\theta_2}$, *then*

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

In other words, the usual rules of exponents work.

Note: For addition and subtraction, it is better to convert back to rectangular form.

Example (Example 6 from Section 8.3)

Let $z_1 = 2e^{i\pi/4}$ and $z_2 = 5e^{i\pi/3}$. Then

$$z_1 z_2 = (2e^{i\pi/4})(5e^{i\pi/3}) = 2 \cdot 5e^{i(\pi/4+\pi/3)} = 10e^{i7\pi/12}$$

and

$$\frac{z_1}{z_2} = \frac{2e^{i\pi/4}}{5e^{i\pi/3}} = \frac{2}{5}e^{i(\pi/4-\pi/3)} = \frac{2}{5}e^{-i\pi/12}.$$

Exponentiation

In the book, De Moivre's Theorem is a cryptic statement about taking a cosine and sine expression to a power. However, using Euler's identity we get the natural relationship:

Theorem

If $z = re^{i\theta}$ *then*

$$z^n = (re^{i\theta})^n = r^n(e^{i\theta})^n = r^ne^{i\theta n}.$$

In other words, the usual rules of exponents work.

Example (Example 7 from Section 8.3)

Find $\left(\frac{1}{2}+\frac{1}{2}i\right)^{10}$.

SOLUTION

Converting to polar form gives $\frac{1}{2}+\frac{1}{2}i=\frac{1}{\sqrt{2}}e^{i\pi/4}$. So by rules of exponents

$$\left(\frac{1}{2}+\frac{1}{2}i\right)^{10}=\left(\frac{1}{\sqrt{2}}e^{i\pi/4}\right)^{10}=\frac{1}{2^5}e^{i10\pi/4}=\frac{1}{32}e^{i5\pi/2}.$$

Since the original question was posed in rectangular form, we should return to that form through Euler's Identity:

$$\frac{1}{32}e^{i5\pi/2}=\frac{1}{32}(\cos 5\pi/2+i\sin 5\pi/2)=\frac{1}{32}i.$$

So

$$\left(\frac{1}{2}+\frac{1}{2}i\right)^{10}=\frac{1}{32}i.$$

Example (Example 9 from Section 8.3)

Find the three cube roots of $z = 2 + 2i$.

SOLUTION

Using cotermial angles, z can be written in polar form in 3 ways:

$$z = 2\sqrt{2}e^{i\pi/4} = 2\sqrt{2}e^{i9\pi/4} = 2\sqrt{2}e^{i17\pi/4}$$

Thus

$$\begin{aligned} z^{1/3} &= (2\sqrt{2}e^{i\pi/4})^{1/3}, \quad (2\sqrt{2}e^{i9\pi/4})^{1/3}, \quad (2\sqrt{2}e^{i17\pi/4})^{1/3} \\ z^{1/3} &= \sqrt{2}e^{i\pi/12}, \quad \sqrt{2}e^{i3\pi/4}, \quad \sqrt{2}e^{i17\pi/12} \end{aligned}$$

The answers in rectangular form are then

$$\begin{aligned} z^{1/3} &= \sqrt{2}(\cos\pi/12 + i\sin\pi/12) \approx 1.366 + 0.366i \\ \text{or} &\quad \sqrt{2}(\cos 3\pi/4 + i\sin 3\pi/4) = -1 + i \\ \text{or} &\quad \sqrt{2}(\cos 17\pi/12 + i\sin 17\pi/12) \approx -0.336 - 1.336i \end{aligned}$$

Write the complex number in polar form with argument θ between 0 and 2π and (a) positive r, (b) negative r

1. $1 + \sqrt{3}i$
2. $-1 + i$
3. -20
4. $\sqrt{3} + i$

Compute the following.

5. $(1 - i\sqrt{3})^5$
6. $(\sqrt{3} - i)^{-10}$
7. $(1 - i)^{-8}$
8. $\sqrt[3]{4\sqrt{3} + 4i}$
9. $\sqrt[5]{32}$
10. $\sqrt[4]{-1}$

Solve for all values of z.

11. $z^8 - i = 0$
12. $z^6 - 1 = 0$

Factor completely.

13. $x^5 - 32$
14. $x^4 + 1$