Math 280 Problems for September 14

Problem 1: Given distinct points $a_1 < a_2 < a_3 < \cdots < a_{100}$ on the real line, determine, with proof, the exact set of real numbers x for which the sum

$$\sum_{i=1}^{100} |x - a_i|$$

takes its minimal value.

Problem 2: Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers, and let a new sequence $q_1, q_2, q_3, \ldots$ be de
ned by $q_1 = a_1$, $q_2 = a_2q_1 + 1$, and $q_n = a_nq_{n+1} + q_{n+2}$ for $n \geq 3$. Prove that no two consecutive q_n's are even.

Problem 3: A function $f(n)$ is defined for all positive integers n as follows: First add the digits of n (in decimal notation) to get a number n_1, say; then add the digits of n_1 to get n_2; continue this process until a single digit number is obtained; that last number (between 1 and 9) is called $f(n)$. Thus, for example, $f(989) = 8$, since $9 + 8 + 9 = 26$, $2 + 6 = 8$. Prove that, for all positive integers n, $f(1234567n) = f(n)$.

Problem 4: Given a nonnegative integer n, let $\widehat{n}$ denote the integer obtained by reversing the digits of n in the standard decimal representation; for example, $\widehat{935} = 539$. Let $f(n) = n + \widehat{n}$, $g(n) = n - \widehat{n}$, and $h(n) = f(g(n))$. For example, if $n = 935$, then $g(n) = 935 - 539 = 396$, and $h(n) = f(396) = 396 + 693 = 1089$ Prove that $h(n) = 1089$ for all three digit integers n whose first digit exceeds the last digit by at least 2.

Problem 5: A polynomial $P(x)$ is known to be of the form

$$P(x) = x^{15} - 9x^{14} + \cdots - 7.$$

where the ellipsis $(\cdots)$ represents unknown intermediate terms. It is also known that all roots of $P(x)$ are integers. Find the roots of $P(x)$.

Problem 6: Does there exist a multiple of 2008 whose decimal representation involves only a single digit (such as 11111 or 22222222)?