Math 280 Problems for September 14

Problem 1: Given distinct points $a_1 < a_2 < a_3 < \cdots < a_{100}$ on the real line, determine, with proof, the exact set of real numbers x for which the sum

$$\sum_{i=1}^{100} |x - a_i|$$

takes its minimal value.

[UIUC 2011 #1] Let $S(x) = \sum_{i=1}^{100} |x - a_i|$ the sum we seek to minimize. By the triangle inequality, we have, for any $i \in \{1, 2, ...100\}$,

$$|a_i - a_{101-i}| = |a_i - x + x - a_{101-i}| \le |a_i - x| + |x - a_{101-i}|$$

Summing over $i = 1, ..., 50$ we get

$$\sum_{i=1}^{50} |a_i - a_{101-i}| \le S(x)$$

The real numbers x that minimize $S(x)$ are exactly those x for which equality holds. However equality holds in the triangle inequality if and only if $a_i - x$ and $x - a_{101-i}$ have the same sign, i.e., if and only if x lies between a_i and a_{101-i}. Hence equality for the sum holds if and only if x lies between a_i and a_{101-i} for each $i = 1, 2, ..., 50$ i.e., if and only if $a_{50} \le x \le a_{51}$. Hence the real numbers that minimize $S(x)$ are exactly those in the interval $a_{50} \le x \le a_{51}$.

Problem 2: Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers, and let a new sequence $q_1, q_2, q_3, \ldots$ be defined by $q_1 = a_1$, $q_2 = a_2 q_1 + 1$, and $q_n = a_n q_{n+1} + q_{n+2}$ for $n \ge 3$. Prove that no two consecutive q_n's are even.

[UIUC 2010 #1] We argue by contradiction. Suppose there exist pairs of consecutive q_n's that are both even. Among these let (q_i, q_{i+1}) be the pair with smallest index i. First note that, if q_1 is even, then $q_2 = a_2 q_1 + 1$ is odd. Thus q_1 and q_2 cannot both be even, so the minimal index i such that q_i and q_{i+1} are both even must be at least 2. By the given recurrence we have $q_{i-1} = q_{i+1} - a_{i+1} q_i$, so q_{i-1} is the difference of two even numbers and therefore must itself be even. Hence (q_{i-1}, q_i) is a pair of consecutive even terms among the q_n's, contradicting the minimality of i. Thus there do not exist consecutive even members of the sequence.

Problem 3: A function $f(n)$ is defined for all positive integers n as follows: First add the digits of n (in decimal notation) to get a number n_1, say; then add the digits of n_1 to get n_2; continue this process until a single digit number is obtained; that last number (between 1 and 9) is called $f(n)$. Thus, for example, $f(989) = 8$, since $9 + 8 + 9 = 26$, $2 + 6 = 8$. Prove that, for all positive integers n, $f(1234567n) = f(n)$.

[UIUC 2010 #2] We use congruences modulo 9. By an extension of the test for divisibility by 9, any positive integer is congruent modulo 9 to the sum of its decimal digits. Since $f(n)$ is obtained by an iteration of the "sum of digits" function, it follows that $f(n)$ satisfies the congruence $f(n) \equiv n \mod 9$. Moreover, since $f(n)$ is in the set $\{1, 2, \ldots, 9\}$, $f(n)$ is uniquely defined by its congruence modulo 9. Thus, to prove the claim, it suffices to show that, for all positive integers n,

$$1234567 \cdot n \equiv n \mod 9$$

But the latter follows from the fact that the number 1234567 is congruent to $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 \equiv 1 modulo 9$.

Problem 4: Given a nonnegative integer n, let $\widehat{n}$ denote the integer obtained by reversing the digits of n in the standard decimal representation; for example, $\widehat{935} = 539$. Let $f(n) = n + \widehat{n}$, $g(n) = n - \widehat{n}$, and $h(n) = f(g(n))$. For example, if $n = 935$, then $g(n) = 935 - 539 = 396$, and $h(n) = f(396) = 396 + 693 = 1089$ Prove that $h(n) = 1089$ for all three digit integers n whose first digit exceeds the last digit by at least 2.

[UIUC 2009 #1] Let n denote an integer of the given form, i.e., $n = a_2 a_1 a_0$ with $a2 \ge a_0 + 2$. Then

$$\begin{aligned}
n &= 100a_2 + 10a_1 + a_0, \\
\widehat{n} &= 100a_0 + 10a_1 + a_2, \\
g(n) &= n - \widehat{n} \\
&= 100 \cdot (a_2 - a_0) - (a_2 - a_0) \\
&= 100 \cdot (a_2 - a_0 - 1) + 10 \cdot 9 + 1 \cdot (10 - a_2 + a_0), \\
\widehat{g(n)} &= 100 \cdot (10 - a_2 + a_0) + 10 \cdot 9 + 1 \cdot (a_2 - a_0 - 1), \\
h(n) &= g(n) + \widehat{g(n)} \\
&= 100 \cdot (10 - 1) + 10 \cdot (9 + 9) + (10 - 1) = 1089,
\end{aligned}$$

as claimed. Note that the given condition on the first and last digits of n, namely $a_2 - a_0 \geq 2$, ensures that the cofficients $a_2 - a_0 - 1$ and $10 - a_2 + a_0$ in the above expressions are integers in the interval $[1, 9]$, so these expressions indeed represent proper decimal expansions.

Problem 5: A polynomial $P(x)$ is known to be of the form

$$P(x) = x^{15} - 9x^{14} + \cdots - 7.$$

where the ellipsis ($\cdots$) represents unknown intermediate terms. It is also known that all roots of $P(x)$ are integers. Find the roots of $P(x)$.

[UIUC 2009 #4] Since $P(x)$ has degree 15, it has 15 roots (counted with multiplicity). Let $r_1, r_2, ..., r_{15}$ denote these roots, which, by assumption, are all integers. Since$P(x)$ has leading term 1, it can be written as

$$P(x) = \prod_{i=1}^{15} (x - r_i).$$

Expanding this product we obtain

$$P(x) = x^{15} + \left(\sum_{i=1}^{15} (-r_i) \right) x^{14} + \cdots + \prod_{i=1}^{15} (-r_i).$$

Comparing this expression with the given form of $P(x)$, we get

$$\sum_{i=1}^{15} (r_i) = 9$$

and

$$\prod_{i=1}^{15} (r_i) = 7.$$

The product forces one of the roots to be 7 or -7, and the remaining 14 roots to be 1 or -1. However, in the case when one of the roots is -7 the sum of all roots can be at most $-7 + 14 = 7$, contradicting the sum requirement. Hence one root must be 7, and the other 14 roots must be 1 or -1. Inspection finds 1 of multiplicity 8 and -1 of multiplicity 6 gives the desired sum. So the roots are

$$7, \underbrace{1, \ldots 1}_{8 \text{ times}}, \underbrace{-1, \cdots - 1}_{6 \text{ times}}$$

Problem 6: Does there exist a multiple of 2008 whose decimal representation involves only a single digit (such as 11111 or 22222222)?

[UIUC 2008 #1] The answer is yes; specifically, we will show that there exists a multiple of 2008 of the form 888 . . . 8. Given a digit $d \in \{1, 2, ..., 9\}$, let $N_{d,k}$ be the number whose decimal representation consists of k digits d. Note that

$$N_{d,k} = d \sum_{i=0}^{k-1} 10^i = \frac{d(10^k - 1)}{9}$$

Thus, a given positive integer m has a multiple of this form if and only if the congruence (*) $d(10^k - 1) \equiv 0 mod 9m$ has a solution k. We apply this with $d = 8$ and $m = 2008$. Then (*) is equivalent to (**) $10^k - 1 \equiv 0 mod 9(2008/8) = 9251$. Since $10^k \equiv 1^k = 1$ mod 9 for any positive integer k, (**) is equivalent to (***) $10^k \equiv 1 \mod 251$. Now, 251 is prime, so by Fermats Theorem, we have $10^{251} 1 \equiv 1 \mod 251$. Thus, (***) holds for $k = 250$, and so the number $N_{8,250} = \underbrace{88 \cdots 8}_{250}$ is divisible by 2008.