Math 280 Problems for October 5

Pythagoras Level

Problem 1: A $3 \times 3 \times 3$ cube is assembled from 27 $1 \times 1 \times 1$ cubes all of whose faces are white. We paint all of the faces of the large cube black, and then disassemble it. A blindfolded man reassembles the large cube from the 27 little cubes. What is the probability that all the faces of the reassembled cube are completely black?

Problem 2: A polynomial of degree 2011 with real coefficients is such that $P(n) = \dfrac{n}{n+1}$ for all integers $n \in \{0, 1, 2, \ldots, 2011\}$. What is the value of $P(2012)$?

Newton Level

Problem 3: Find all polynomials $P(x)$ such that $P(2x) = P'(x) \cdot P''(x)$ for all $x \in \mathbb{R}$.

Problem 4: Let $y = x^{1/x}$ for $x > 0$. Find the intervals on which $y(x)$ is monotonic, and on each such interval, find its range.

Wiles Level

Problem 5: A vector $\vec{v} = (x, y, z) \in \mathbb{R}^3$ is integral if each component is an integer. Prove that if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are mutually orthogonal integral vectors with the same length L, then L is an integer.

Problem 6: Consider a binary operation $*$ on a set S, that is, for all $a, b \in S$, $a * b$ is in S. Prove that if for all $a, b \in S$, $(a * b) * a = b$, then for all $a, b \in S$, $a * (b * a) = b$. (Obviously you cannot assume $*$ is associative.)