Math 280 Problems for September 6

Pythagoras Level

#1 If n is a positive integer, let $r_b(n)$ denote the number obtained by reversing the order of the base-b digits of n. For example, $r_9(317) = r_9(382_9) = 283_9 = 237$ and $r_5(110) = r_5(410_5) = 14_5 = 9$. Fix a base b. For how many two b-digit positive integers n is the sum of n and $r_b(n)$ a perfect square?

#2 Alan and Barbara play a game in which they take turns filling entries of an initially empty 2010×2010 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

Newton Level

#3 Let $g(x) = e^{x^2}$. Find $g^{(2010)}(0)$, i.e., the 2010th derivative of g evaluated at $x = 0$.

#4 Consider the sequence $a_1 = 1$, $a_{n+1} = \sqrt{2}^{a_n}$. Show that $\lim_{n\to\infty} a_n$ exists and compute its value.

Wiles Level

#5 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

#6 Consider the infinite series

$$\frac{1}{1} + \frac{1}{10} + \frac{1}{11} + \frac{1}{100} + \frac{1}{101} + \frac{1}{110} + \frac{1}{111} + \cdots$$

whose terms are the reciprocals of all positive integers which have only 0's and 1's as digits (taken in the implied order). Does this series converge or diverge? Prove your answer.