Math 280 Problems for September 27

Pythagoras Level

Problem 1: Find all pairs of real numbers (x, y) satisfying the system of equations

$$\frac{1}{x} + \frac{1}{2y} = (x^2 + 3y^2)(3x^2 + y^2)$$
$$\frac{1}{x} - \frac{1}{2y} = 2(y^4 - x^4).$$

Problem 2: Suppose n fair 6-sided dice are rolled simultaneously. What is the expected value of the score on the highest valued die?

Newton Level

Problem 3: Let f be a continuous function on $[0, 1]$, differentiable on $(0, 1)$, and such that $f(1) = 0$. Show that for some $c \in (0, 1)$,

$$\frac{f(c)}{c} = -f'(c).$$

Problem 4: Let $f : [0, 1) \to \mathbb{R}$ be a continuous, strictly increasing function, such that

$$(f(x))^3 = \int_0^x t(f(t))^2 dt$$

for every $x \geq 0$. Show that for every $x \geq 0$ we have $f(x) = \frac{x^2}{6}$.

Wiles Level

Problem 5: Given that a and b are real numbers satisfying $a^3 - 3ab^2 = 39$ and $b^3 - 3a^2b = \sqrt{487}$, determine $a^2 + b^2$.

Problem 6: Let f be a nonconstant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.