Math 280 Problems for October 4

Pythagoras Level

Problem 1: Let $f(n) = 25^n - 72n - 1$. Determine, with proof, the largest integer M such that $f(n)$ is divisible by M for every positive integer n.

Problem 2: Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \dots}}}.$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers.
Hint: $(x^2 - ax + 1)(x^2 + ax + 1) = (x^2)^2 - (a^2 - 2)x^2 + 1$.

Newton Level

Problem 3: Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx.$$

Problem 4: Let $n \geq 2$ be an integer and define $f(x) = 1 - x^n$. For each $t \in (0,1)$, let A_t denote the area of the triangle in the first quadrant formed by the x-axis, y-axis, and the tangent line to $f(x)$ at $x = t$. Find $t \in (0,1)$ so that A_t is a minimum.

Wiles Level

Problem 5: Let $n \geq 1$. Pick at random a function

$$f : \{1, ..., n\} \to \{1, 2, 3\}$$

What is the probability P of f not being onto (surjective)?

Problem 6: Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.