Math 280 Problems for October 18

Pythagoras Level

Problem 1: Given that

$$\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}} = 3,$$

compute

$$r^3 + \frac{1}{r^3}.$$

Problem 2: The positive numbers r and t are related by the fact that if the radius r of a circle is increased by t, the area is doubled. Express r as a function of t.

Newton Level

Problem 3: Find a real number r such that if an integer $n \geq r$, then

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n^2} \geq 2008$$

Problem 4: Let $f(x) = \int_0^x e^{-t^2}\, dt$. Given that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$, evaluate

$$\int_0^{\infty} e^{-x^2+f(x)}\, dx.$$

Wiles Level

Problem 5: A card shuffling machine always rearranges cards in the same way relative to the order in which they are given to it. The thirteen spades arranged in the order

$$A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$$

are put into the machine, shuffled, and then the shuffled cards are put into the machine and shuffled again. If at this point the order of the cards is

$$3, K, 10, 2, Q, 9, 4, J, 8, 6, 7, A, 5,$$

what was the order of the cards after the first shuffle?

Problem 6: For positive numbers r, let $F(r)$ denote the fractional part of r; i.e., $F(r) = r - \lfloor r \rfloor$. Thus, e.g., $F(8/3) = 2/3$. Find a positive number r such that

$$F(r) + F\left(\frac{1}{r}\right) = 1$$