p-Adic Egyptian Fractions

Contents

1 Introduction

An Egyptian fraction is a sum of distinct positive unit fractions. In many cases a given fraction has several representations, for example

$$\frac{5}{121} = \frac{1}{25} + \frac{1}{757} + \frac{1}{763309} + \frac{1}{873960180913} + \frac{1}{1527612795642093418846225}$$

or more simply, $\frac{5}{121} = \frac{1}{33} + \frac{1}{99} + \frac{1}{1089}$. The former is obtained from the greedy algorithm while the latter comes from performing the greedy 11-adic algorithm that we will introduce in Section 4.

2 Traditional Egyptian Fractions and Greedy Algorithm

Proposition 1 (Classical Division Algorithm)**.** For all positive integers $a, b \in \mathbb{Z}$ there exist unique positive integers q and r such that $b = aq - r$ with r strictly less than a.

Proof. This proof is similar to the standard proof of the original classical division algorithm. Let a and b be positive integers. To show existence, consider the set $S = \{an - b \geq 0 : n \in \mathbb{Z}\}$. Clearly there is at least one nonnegative entry since a large multiple of a must exceed b. Among these entries there is a least called r that's guaranteed by the Well-Ordering Principle. Then by definition for some integer, denoted q, we have $r = aq - b$ or $b = aq - r$ as desired.

As with the original classical algorithm we require $0 \leq r < a$. The first inequality is a given since it's an element of S. To prove that the second holds, suppose for contradiction that $r \geq a$. Then $r = aq - b \geq a$ implies $r - a = a(q-1) - b \geq 0$ which shows that $r > r - a \in S$ which contradicts the minimality of r.

Like the classical division algorithm we obtain uniqueness. Suppose that $b = aq - r = aq' - r'$. Subtracting gives $a(q - q') = r - r'$ so $a \mid r - r'$. Which is impossible since $|r - r'| < a$ unless $r - r' = 0$ and so $r = r'$ as needed. Then, $aq = aq'$ and $q = q'$ as well. □

Definition 1 (The Greedy Algorithm)**.** The Greedy Algorithm applied to $\dfrac{a}{b}$ starts by using the modified division algorithm to find the unique q_1 and r_1 such that

$$b = aq_1 - r_1.$$

Next, we apply the modified division algorithm to bq_1 and r_1. Continuing we have,

$$\begin{aligned} b &= aq_1 - r_1 \\ bq_1 &= r_1 q_2 - r_2 \\ &\vdots \\ bq_1 q_2 q_3 \dots q_{n-2} &= r_{n-2} q_{n-1} - r_{n-1} \\ bq_1 q_2 q_3 \dots q_{n-2} q_{n-1} &= r_{n-1} q_n - 0 \qquad \text{with } r_n = 0. \end{aligned}$$

The algorithm eventually terminates since the modified division algorithm requires each remainder to be less than the previous and therefore yields a decreasing sequence of positive integers.

Notice that rearranging the terms from the algorithm yields,

$$\begin{aligned}
\frac{a}{b} &= \frac{1}{q_1} + \frac{r_1}{bq_1} && (\star) \\
\frac{r_1}{bq_1} &= \frac{1}{q_2} + \frac{r_2}{bq_1q_2} \\
&\vdots \\
\frac{r_{n-2}}{bq_1q_2q_3 \dots q_{n-2}} &= \frac{1}{q_{n-1}} + \frac{r_{n-1}}{bq_1q_2q_3 \dots q_{n-2}q_{n-1}} \\
\frac{r_{n-1}}{bq_1q_2q_3 \dots q_{n-2}q_{n-1}} &= \frac{1}{q_n}
\end{aligned}$$

which shows that the sum of the reciprocals of the q_i is equal to $\frac{a}{b}$ i.e.

$$\frac{a}{b} = \sum_{i=1}^{n} \frac{1}{q_i}$$

To be an Egyptian fraction expansion each term must be distinct. Indeed, $q_i > q_{i-1}$ since $q_1 = \lceil \frac{b}{a} \rceil$ and $q_2 = \lceil \frac{bq_1}{r_1} \rceil = \lceil \frac{b}{r_1/q_1} \rceil > \lceil \frac{b}{a} \rceil$, the last inequality follows from $\frac{r_1}{q_1} < a$.

Example 1. Performing the Classical Greedy Algorithm on $\dfrac{5}{11}$ gives the following expansion,

$$\frac{5}{11} = \frac{1}{3} + \frac{1}{9} + \frac{1}{99}.$$

3 Set-up

Definition 2 (Order)**.** The order of some integer n with respect to a prime p, denoted $\operatorname{ord}_p(n)$, is the power of p that exactly divides n. We define

$$\operatorname{ord}_p\left(\frac{a}{b}\right) = \operatorname{ord}_p(a) - \operatorname{ord}_p(b).$$

Definition 3 (Unit part)**.** Let $r = \frac{a}{b} \in \mathbb{Q}$ and p prime. The unit part of r, denoted $\widehat{r}$ is given by

$$\widehat{r} := rp^{-\mathrm{ord}_p(r)}.$$

Note that $\mathrm{ord}_p\left(\widehat{a}\right) = 0$ and $\widehat{a}$ and p are relatively prime. Throughout we will write $\mathrm{ord}_p\left(a\right) = \alpha$ and $\mathrm{ord}_p\left(b\right) = \beta$.

Proposition 2. If $a, b \in \mathbb{Q}$ then $\mathrm{ord}_p\left(ab\right) = \mathrm{ord}_p\left(a\right) + \mathrm{ord}_p\left(b\right)$

Proof. Using units we write $a = \widehat{a}p^{\alpha}$ and $b = \widehat{b}p^{\beta}$. Then

$$\mathrm{ord}_p\left(ab\right) = \mathrm{ord}_p\left(\widehat{a} \cdot \widehat{b} \cdot p^{\alpha+\beta}\right) = \alpha + \beta.$$

□

Lemma 1. If $a, b \in \mathbb{Q}$ and p prime then $\mathrm{ord}_p\left(a+b\right) \geq \min\{\mathrm{ord}_p\left(a\right), \mathrm{ord}_p\left(b\right)\}$. We have equality when $\mathrm{ord}_p\left(a\right) \neq \mathrm{ord}_p\left(b\right)$.

Proof. Taking $a, b \in \mathbb{Q}$ we assume without loss of generality that $\beta \leq \alpha$. Using units we write $a = \widehat{a}p^{\alpha}$ and $b = \widehat{b}p^{\beta}$.

$$\begin{aligned}
\mathrm{ord}_p\left(a+b\right) &= \mathrm{ord}_p\left(\widehat{a}p^{\alpha} + \widehat{b}p^{\beta}\right) \\
&= \mathrm{ord}_p\left(p^{\beta}\left(\widehat{a}p^{\alpha-\beta} + \widehat{b}\right)\right) \\
&= \beta + \mathrm{ord}_p\left(\widehat{a}p^{\alpha-\beta} + \widehat{b}\right). && \text{Proposition 2}
\end{aligned}$$

Should $\alpha = \beta$, we have

$$\mathrm{ord}_p\left(a+b\right) = \beta + \mathrm{ord}_p\left(\widehat{a} + \widehat{b}\right) \geq \min\left\{\alpha, \beta\right\}$$

In the case where $\alpha \neq \beta$ we have that $\alpha - \beta$ is positive. Then the order of $\widehat{a}p^{\alpha-\beta} + \widehat{b}$ is zero and

$$\mathrm{ord}_p\left(a+b\right) = \min\{\alpha, \beta\}.$$

□

4 p-Greedy Algorithm

Theorem 2 (p-Division Algorithm)**.** Let $a, p \in \mathbb{Z}$ with p prime and $b \in \mathbb{Z}[\frac{1}{p}] = \{mp^k \mid m, k \in \mathbb{Z}\}$. Then, there exist unique $r \in \mathbb{Z}$ and $q \in \mathbb{Z}[\frac{1}{p}]$ such that $b = aq - r$ with $0 \leq r < ap$ and $\mathrm{ord}_p(r) > \mathrm{ord}_p(a)$.

Proof.

Existence Let $a, b \in \mathbb{Z}$ and p prime. Since $\widehat{a}$ and p are relatively prime any power of p has an inverse mod $\widehat{a}$. Take $0 \leq \overline{r} < \widehat{a}$ such that $\overline{r} \equiv -\widehat{b}p^{\beta-\alpha-1} \mod \widehat{a}$. For some integer m we can write,

$$\begin{aligned} \overline{r}p^{-\beta+\alpha+1} - (-\widehat{b}) &= \widehat{a}m \\ \overline{r}p^{\alpha+1} + \widehat{b}p^{\beta} &= \widehat{a}mp^{\beta} \\ \widehat{b}p^{\beta} &= \widehat{a}p^{\alpha}mp^{\beta-\alpha} - \overline{r}p^{\alpha+1} \\ b &= aq - r \end{aligned}$$

with $q = mp^{\beta-\alpha}$ and $r = \overline{r}p^{\alpha+1}$.

Uniqueness Given $a, b \in \mathbb{Z}$ and p prime suppose that there exist corresponding q_1, r_1, q_2, r_2 such that $b = aq_1 - r_1 = aq_2 - r_2$. Then,

$$a(q_1 - q_2) = r_1 - r_2 \tag{1}$$

By (1) we have that $r_1 \equiv r_2 \mod \widehat{a}$ and $r_1 \equiv r_2 \mod p^{\alpha+1}$ since by assumption $\mathrm{ord}_p(r_i) > a$. Therefore, $r_1 \equiv r_2 \mod ap$. Thus, $r_1 = r_2$ or only one is between 0 and ap; so $r_1 = r_2$ and $q_1 = q_2$ and we have uniqueness as desired.

□

Note, it's possible that $\overline{r} \neq \widehat{r}$.

The value of $\mathrm{ord}_p(\overline{r})$ can vary depending on the choice of prime. Consider the case where $a = 23$ and $b = 60$. If we take $p = 5$ then $a = 23 \cdot 5^0, b = 12 \cdot 5^1$. Then,

$$\overline{r} \equiv -(12)(5)^{1-0-1} \equiv 11 \pmod{23}$$

and $\mathrm{ord}_5(\overline{r}) = 0$. However, if $p = 3$ then $a = 23 \cdot 3^0, b = 20 \cdot 3^1$. Then,

$$\overline{r} \equiv -(20)(3)^{1-0-1} \equiv 3 \pmod{23}$$

and the order of $\overline{r}$ is 1.

Definition 4. A **jump** occurs when after performing the p-Division Algorithm we have $\operatorname{ord}_p(\overline{r}) \geq 1$.

Lemma 3. If $p > \widehat{a}$ then $\overline{r} = \widehat{r}$, i.e. there is no jump.

Proof. Simply note that $\overline{r}$ is taken mod $\widehat{a}$; hence $\overline{r} < \widehat{a} < p$. $\square$

Definition 5 (p-Greedy Algorithm)**.** Let $a, b, p \in \mathbb{Z}$ with p prime. Theorem 2 gives unique $q_1 \in \mathbb{Z}[\frac{1}{p}]$ and $r_1 \in \mathbb{Z}$ such that $b = aq_1 - r_1$. Repeating on bq_1 and r_1 gives,

$$\begin{aligned}
b &= aq_1 - r_1 \\
bq_1 &= r_1 q_2 - r_2 \\
&\vdots \\
bq_1 b_2 \cdots q_{n-2} &= r_{n-2} q_{n-1} - r_{n-1} \\
bq_1 q_2 \cdots q_{n-1} &= r_{n-1} q_n - 0 && \text{until } r_n = 0
\end{aligned}$$

Note by **??**($\star$) we again get $\frac{a}{b} = \sum \frac{1}{q_i}$.

Example 2. Consider the fraction $\frac{12}{5}$. Performing the Greedy 3-Adic Algorithm yields,

$$\begin{aligned}
5 &= 12\left(\frac{8}{3}\right) - 27 && \overline{r}_1 = 3 \\
5\left(\frac{8}{3}\right) &= 27\left(\frac{40}{81}\right) - 0
\end{aligned}$$

and

$$\frac{12}{5} = \frac{3}{8} + \frac{81}{40}.$$

In the classical case typically the q's are any integer greater than one; these come from expanding a fraction whose size is less than one. We get $q = 1$ only if the fraction being expanded is greater than one. (The classical greedy algorithm always takes the next largest unit fraction, including 1) The p-Adic q's from the above example look very different but are in fact typical in that their order is decreasing and we will see later that they are obtained from expanding fractions whose order is nonnegative. Fractions with negative order lead to the atypical p-Adic case where $q = p$ as shown below. Even though the p-Adic case deals with order, it should be noted that as elements of the p-Adic field $\mathbb{Q}_p$, fractions with negative order are considered greater than one, and $q = p$ leads to the term $1/p$ appearing in the expansion which is analogous to $1/1$ appearing in a classical expansion since under the p-Adic norm $1/p$ has size p.

Example 3. Consider the fraction $\frac{85}{126} = \frac{17 \cdot 5}{3^2 \cdot 7 \cdot 2}$. Performing the Greedy 7-Adic Algorithm yields,

$$\begin{aligned} 126 &= 85(7) - 469 & \bar{r}_1 &= 67 \\ 126 \cdot 7 &= 469(7) - 2401 & \bar{r}_2 &= 49 \\ 126 \cdot 7 \cdot 7 &= 2401\left(\frac{18}{7}\right) - 0 \end{aligned}$$

and

$$\frac{85}{126} = \frac{1}{7} + \frac{1}{7} + \frac{7}{18}.$$

Theorem 4. The p-Greedy Algorithm terminates in a finite number of steps.

Proof. Note that $\bar{r}_k$ is taken mod $\widehat{r}_{k-1}$ with representatives in $\{0, 1, 2, \ldots, \widehat{r}_{k-1} - 1\}$ and therefore, $\widehat{r}_k \leq \bar{r}_k < \widehat{r}_{k-1}$. Since $\widehat{r}_k$ is nonnegative, this is a decreasing sequence of natural numbers that must end with zero. Eventually $\widehat{r_n}$ and r_n are identically zero and the algorithm terminates. □

Corollary 5. The p-Greedy Algorithm terminates in at most $\widehat{a}$ steps. This follows from the fact that the sequence of the unit parts of the remainders $\widehat{r_1}, \widehat{r_2}, \ldots, 0$ is defined so that $\widehat{r_i}$ is taken mod $\widehat{r}_{i-1}$ with $\widehat{r_0} := \widehat{a}$.

Definition 6 (Pure p-Egyptian Fraction)**.** For a fixed prime p, if

$$\frac{a}{b} = \sum_{i=1}^{n} \frac{p^{\ell_i}}{m_i}$$

with ℓ_i strictly increasing nonnegative and m_i positive relatively prime to p then the sum is a p-Egyptian fraction expansion for $\frac{a}{b}$.

Lemma 6 (Classification)**.** Let $a, b, p \in \mathbb{Z}$ with p prime. Suppose that $\ell = \operatorname{ord}_p\left(\frac{a}{b}\right) \leq -1$. Let q denote the quotient from the p-adic division algorithm and q' the quotient from the classical division algorithm on $\frac{ap}{b}$. Then, $q = pq'$.

Proof. Since $\ell \le -1$ we can write

$$\frac{a}{b} = \frac{\widehat{a}}{\widehat{b}p^{-\ell}} \qquad\qquad \frac{ap}{b} = \frac{\widehat{a}}{\widehat{b}p^{-\ell-1}}.$$

The p-adic division algorithm gives $\overline{r}$ and q such that,

$$\begin{aligned}
\widehat{b}p^{-\ell} &= \widehat{a}q - \overline{r}p \\
\overline{r} &\equiv -\widehat{b}p^{-\ell-1} \pmod{\widehat{a}} \\
q &= \frac{\widehat{b}p^{-\ell} + \overline{r}p}{\widehat{a}}.
\end{aligned}$$

Applying the modified classical division algorithm yields q' and r' such that,

$$\begin{aligned}
\widehat{b}p^{-\ell-1} &= \widehat{a}q' - r' \\
r' &= \widehat{a}q' - \widehat{b}p^{-\ell-1} \\
r' &\equiv -\widehat{b}p^{-\ell-1} \equiv \overline{r} \pmod{\widehat{a}} \\
r' &= \overline{r} \qquad \text{(smallest nonnegative representatives always used)} \\
q' &= \frac{\widehat{b}p^{-\ell-1} + r'}{\widehat{a}}.
\end{aligned}$$

Finally,

$$q'p = \frac{\widehat{b}p^{-\ell} + rp}{\widehat{a}} = \frac{\widehat{b}p^{-\ell} + \overline{r}p}{\widehat{a}} = q.$$

□

Corollary 7. If $\mathrm{ord}_p\left(\frac{a}{b}\right) \le -1$ and if no jumps occur than each term of the p-Adic Greedy Algorithm for $\frac{a}{b}$ is equal to the corresponding term in the classical Greedy Algorithm on $\frac{ap}{b}$ divided by p.

For example, consider the fraction $\frac{a}{b} = \frac{5}{121} = \frac{5^1}{11^2}$. If we take $p = 11$ then $\mathrm{ord}_p\left(\frac{a}{b}\right) = -2$. Applying the 11-Adic Greedy Algorithm to $\frac{5}{121}$ yields $\overline{r}_1 = 4, \overline{r}_2 = 3, \overline{r}_3 = 0$ all of order zero. Using the q_i we can write,

$$\frac{5}{121} = \frac{1}{33} + \frac{1}{99} + \frac{1}{1089}.$$

Recall example 1 gave $\frac{ap}{b} = \frac{5\cdot 11}{121} = \frac{1}{3} + \frac{1}{9} + \frac{1}{99}$.

Note, the restriction on jumps is sufficient but not necessary, for an example use $\frac{22}{45}$ with $p = 3$.

Lemma 8. Suppose that $\text{ord}_p\left(\frac{a}{b}\right) \geq 0$, then the quotient from the division algorithm is not p.

Proof. If $\text{ord}_p\left(\frac{a}{b}\right) > 0$ then $\text{ord}_p(q) = \beta - \alpha \neq 0$. Suppose $\text{ord}_p\left(\frac{a}{b}\right) = 0$. For sake of a contradiction, suppose that $q = p$. Since $q = mp^{\beta-\alpha}$ and the order is zero we have that $q = m = p$ and

$$\overline{r}p + \widehat{b} = \widehat{a}m = \widehat{a}p$$

however this is impossible since

$$\text{ord}_p\left(\overline{r}p + \widehat{b}\right) = 0 \neq 1 = \text{ord}_p(\widehat{a}p).$$

□

Theorem 9. Let $a, b, p \in \mathbb{Z}$ with p prime. Suppose that $\text{ord}_p\left(\frac{a}{b}\right) \leq -1$ and $\lfloor \frac{ap}{b} \rfloor \geq 1$, then dividing b by a using the p-Adic Division Algorithm yields $q = p$ and

$$\frac{a}{b} = \frac{1}{p} + \frac{r}{bp}.$$

Proof. The fraction $\frac{ap}{b} \geq 1$ and therefore its Classical Greedy Egyptian Fraction Expansion begins with 1. By assumption $\text{ord}_p\left(\frac{a}{b}\right) \leq -1$ so by Lemma 6 the quotient from using the p-Adic Divison Algorithm to divide b by a is p. □

Theorem 10. If $a, p \in \mathbb{Z}$ with p prime and $b \in \mathbb{Z}[\frac{1}{p}]$ and $0 \leq \text{ord}_p\left(\frac{a}{b}\right)$ then applying the p-Adic Division algorithm to divide b by a yields q and r such that

$$\text{ord}_p(bq) \leq \text{ord}_p(b) \leq \text{ord}_p(a) < \text{ord}_p(r).$$

Proof. The p-Adic Division algorithm says that $r = \overline{r}p^{\text{ord}_p(a)+1}$ which allows us to write

$$\text{ord}_p(a) < \text{ord}_p(a) + 1 \leq \text{ord}_p(r)$$

By hypothesis we already have $\text{ord}_p(b) \leq \text{ord}_p(a)$. Therefore, it remains to show that $\text{ord}_p(q) \leq 0$. Drawing on the p-Division algorithm again we have,

$$\begin{aligned}
\text{ord}_p(q) &= \text{ord}_p(b + r) - \text{ord}_p(a) \\
&= \text{ord}_p(b) - \text{ord}_p(a) && \text{since } \text{ord}_p(b) < \text{ord}_p(r) \\
&\leq 0.
\end{aligned}$$

Hence, $\text{ord}_p(bq) = \text{ord}_p(b) + \text{ord}_p(q) \leq \text{ord}_p(b)$. □

Theorem 11. Let $a, b, p \in \mathbb{Z}$ with p prime. Then performing the p-Adic Greedy Algorithm yields quotients such that $q_i = p$ for $i \leq K$ where K is the least K such that one of the following is true:

$$\left\lfloor \frac{ap - bK}{bp} \cdot p \right\rfloor \leq 1 \tag{2}$$

$$\operatorname{ord}_p \left(\frac{ap - bK}{bp} \right) \geq 0. \tag{3}$$

Proof. First note that K always exists since the fraction in condition 2 is decreasing. For $0 \leq i < K$ we have that the negations of 2 and 3 are satisfied. Should $K = 0$ then either $\lfloor \frac{ap}{b} \rfloor \geq 1$ in which case the quotient from the Greedy Expansion isn't 1 and therefore q_1 from the p-Adic Greedy Algorithm isn't p or $\operatorname{ord}_p \left(\frac{a}{b} \right) \geq 0$ and by Lemma 8 $q_1 \neq p$.

Suppose that $K \geq 1$. Recall that at each step of the p-Adic Greedy Algorithm we perform the p-Adic Division Algorithm on

$$\frac{r_K}{bq_1 \cdots q_K} = \frac{a}{b} - \frac{K}{p} = \frac{ap - bK}{bp}.$$

Note that K is the first integer such that $\frac{ap-bK}{bp}$ fails to satisfy the hypotheses of theorem 9. Subsequent remainders are either strictly less than $\frac{ap-bK}{bp}$ or their orders are greater by Lemma 10. $\square$

5 p-Egyptian Traditional

In both the classical and the p-adic case, repeated application of a modified division algorithm yielded the following sum,

$$\frac{a}{b} = \sum_{i=1}^{n} \frac{1}{q_i}.$$

The classical q_i are always integers so the sum is an Egyptian fraction expansion. The p-adic case yields q_i that force powers of p to appear in the numerator of each term. Even though this sum isn't immediately an Egyptian fraction expansion, it becomes one once you divide out by the highest power of p present among the terms. Of course it's no longer an Egyptian fraction expansion for $\frac{a}{b}$ but it is a valid expansion for another rational, namely $\dfrac{a}{bp^{-\mathrm{ord}_p(q_n)}}$.

It's possible in some cases to reverse this process by starting with a fraction $\frac{a}{b}$ and for each prime that makes up b, remove it and find the corresponding p-adic Egyptian fraction expansion for what remains. A fraction works if the desired power k of p in the denominator is greater than or equal to the order of the last term in the expansion so that dividing out doesn't leave any powers of p behind. The order of $1/q_n$ sets the lowest power of p that is allowed in the denominator.

Example 4. Suppose that we want an Egyptian fraction expansion for

$$\frac{13}{189} = \frac{13}{3^3 7}.$$

From the primes in the denominator we see that we can either perform the 3-Adic algorithm on $\dfrac{13}{7}$ or the 7-Adic algorithm on $\dfrac{13}{27}$. In the 3-Adic case we will be successful if the final term in the expansion has no more than 3 powers of $p = 3$ in the numerator, while the 7-Adic case would require that the final term have order at most one.

The 7-Adic case yields,

$$\frac{13}{27} = \frac{1}{8} + \frac{7}{33} + \frac{343}{2376}.$$

Dividing by 7 we find that

$$\frac{13}{189} = \frac{1}{56} + \frac{1}{33} + \frac{49}{2376}$$

which won't work for our purposes since there are still terms of positive order in the expansion.

The 3-Adic case yields,

$$\frac{13}{7} = 1 + \frac{3}{8} + \frac{27}{56}$$

which when divided by 3^3 yields,

$$\frac{13}{189} = \frac{1}{27} + \frac{1}{72} + \frac{1}{56}$$

which is an Egyptian fraction expansion that's significantly more balanced than the one obtained from the classical greedy algorithm:

$$\frac{13}{189} = \frac{1}{15} + \frac{1}{473} + \frac{1}{446985}.$$

The kinds of fractions that can be expanded in this way depend on the order of q_n.

Corollary 12. Inspecting subsequent steps of the p-Greedy algorithm we see that,

$$\mathrm{ord}_p\left(q_k\right) = \mathrm{ord}_p\left(bq_1q_2\ldots q_{k-1}\right) - \mathrm{ord}_p\left(r_{k-1}\right).$$

Proof. Since $q_k r_{k-1} = bq_1\cdots q_{k-1} + r_k$ and $\mathrm{ord}_p\left(bq_1\cdots q_{k-1}\right) < \mathrm{ord}_p\left(r_k\right)$ by Lemma 1 $\mathrm{ord}_p\left(q_k\right) = \mathrm{ord}_p\left(\frac{bq_1\cdots q_{k-1}}{r_{k-1}}\right) = \mathrm{ord}_p\left(bq_1q_2\ldots q_{k-1}\right) - \mathrm{ord}_p\left(r_{k-1}\right)$. $\square$

Theorem 13. Let $a, b, p \in \mathbb{Z}$ with p prime and $\mathrm{ord}_p\left(b\right) = \beta, \mathrm{ord}_p\left(a\right) = \alpha$. Then,

$$\mathrm{ord}_p\left(q_n\right) = 1 + 2^{n-1}\left(\beta - \alpha - 1 - \sum_{i=1}^{n-1}\frac{\mathrm{ord}_p\left(\overline{r}_i\right)}{2^i}\right)$$

with q_n and $\overline{r}_i$ defined as per the p-Greedy algorithm in Definition 5.

Proof. We proceed by induction on n.

Base Case: $n = 1$. By Theorem 2 the order of q_1 is $\beta - \alpha = 1 + 2^0(\beta - \alpha - 1)$.

Inductive Case: Suppose for some $n \geq 1$ that

$$\mathrm{ord}_p\left(q_n\right) = 1 + 2^{n-1}\left(\beta - \alpha - 1 - \sum_{i=1}^{n-1}\frac{\mathrm{ord}_p\left(\overline{r}_i\right)}{2^i}\right).$$

By corollary 12 we already have that

$$\mathrm{ord}_p\left(q_{n+1}\right) = \mathrm{ord}_p\left(bq_1\ldots q_n\right) - \mathrm{ord}_p\left(r_n\right).$$

Then,

$$\begin{aligned}\mathrm{ord}_p\left(q_{n+1}\right) &= \mathrm{ord}_p\left(bq_1\ldots q_n\right) + \mathrm{ord}_p\left(r_{n-1}\right) - \mathrm{ord}_p\left(r_{n-1}\right) - \mathrm{ord}_p\left(r_n\right)\\ &= \mathrm{ord}_p\left(bq_1\ldots q_{n-1}\right) + \mathrm{ord}_p\left(q_n\right) + \mathrm{ord}_p\left(r_{n-1}\right) - \mathrm{ord}_p\left(r_{n-1}\right) - \mathrm{ord}_p\left(r_n\right)\end{aligned}$$

$$
\begin{aligned}
&= 2 \cdot \text{ord}_p\left(q_n\right) + \text{ord}_p\left(r_{n-1}\right) - \text{ord}_p\left(r_n\right) && \text{Corollary 12} \\
&= 2 \cdot \text{ord}_p\left(q_n\right) + \text{ord}_p\left(r_{n-1}\right) - \text{ord}_p\left(\overline{r}_n\right) - \text{ord}_p\left(r_{n-1}\right) - 1 && \text{Since } r = \overline{r}p^{\alpha+1} \\
&= 2 + 2^n\left(\beta - \alpha - 1 - \sum_{i=1}^{n-1}\frac{\text{ord}_p\left(\overline{r}_i\right)}{2^i}\right) - \text{ord}_p\left(\overline{r}_n\right) - 1 \\
&= 1 + 2^n\left(\beta - \alpha - 1 - \sum_{i=1}^{n-1}\frac{\text{ord}_p\left(\overline{r}_i\right)}{2^i}\right) - \frac{2^n\text{ord}_p\left(\overline{r}_n\right)}{2^n} \\
&= 1 + 2^n\left(\beta - \alpha - 1 - \sum_{i=1}^{n}\frac{\text{ord}_p\left(\overline{r}_i\right)}{2^i}\right).
\end{aligned}
$$

Thus the order of q_n is given by,

$$
\text{ord}_p\left(q_n\right) = 1 + 2^{n-1}\left(\beta - \alpha - 1 - \sum_{i=1}^{n-1}\frac{\text{ord}_p\left(\overline{r}_i\right)}{2^i}\right).
$$

□

Corollary 14. If $\text{ord}_p\left(\frac{a}{b}\right) = 0$ then

$$
\text{ord}_p\left(q_n\right) = 1 - 2^{n-1} - \sum_{i=1}^{n-1} 2^{n+i-1}\text{ord}_p\left(\overline{r}_i\right).
$$

6 Conclusion

- The classical algorithm can handle irrational numbers since since $q = \lceil b/a \rceil$ where $b = 1$ and a irrational. Although p-Adic numbers have an associated norm, there are many rationals that satisfy $|x|_p = 1$ e.g. all rationals of order 0 so it's difficult to pick the "correct" one. There is also no clear way to define what it means for one to "mod out" by $\widehat{a}$ if it's irrational.

- The classical greedy algorithm for egyptian fractions always takes the next largest unit fraction. Therefore, if a term is deleted the expansion for the remaining terms will simply be the original expansion with the one term deleted. The same appears to be true for the p-Adic algorithm, however if one defines "next largest unit fraction" as the next $1/q$ (in the classical case this corresponds to the next largest by absolute value) than all you know is that some $1/q$ is "less" than a/b, however there's no way to test this unless one actually carries out the algorithm. Of course, one cannot expect a perfect analogy to always exist.

- The unit part of q_n can be roughly approximated by $\left(\frac{\widehat{b}}{\widehat{a}}\right)^{2^{n-1}} 2^{n-1}$.

  Assuming that $\mathrm{ord}_p\left(\frac{a}{b}\right) = 0$ and that $p > \widehat{a}$ allows us to determine the order of q_n exactly and we have that the $1/q$ terms grow as

$$\frac{p^{2^{n-1}-1}}{\left(\frac{\widehat{b}}{\widehat{a}}\right)^{2^{n-1}} 2^{n-1}}$$

  hence if $\widehat{b}/\widehat{a} \approx p/2$ we obtain particularily balanced terms.

- The p-Adic algorithm yields an expansion with at most $\widehat{a}$ terms. It remains to be seen if these examples always exist.

- It would also pay to investigate the average lengths of expansions and compare it to the average length of sequences if $\widehat{r}_i$ is assumed to be a random integer mod $\widehat{r}_{i-1}$.