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THE NATURAL CHAIN OF BINARY ARITHMETIC OPERATIONS AND GENERALIZED DERIVATIVES

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Abstract. [redacted]

Introduction

Nearly every culture in the world has discovered and made use of the natural numbers and their extensions the integers, the rationals, and beyond. The natural operation of counting gives rise quite naturally to the operations of addition and multiplication. Moreover, these operations are intimately related. Indeed, multiplication is usually defined in terms of addition and is considered universally as a kind of shorthand for addition. Perhaps because of this, multiplication and addition have a natural relationship expressed by the distributive law. This law binds these two operations together and endows the set of integers with the algebraic structure of a ring.

The logarithm and exponential maps are compatible with the intimate relationship between multiplication and addition. The logarithm maps products into sums in such a way that the multiplicative structure of the positive real numbers and the additive structure of all the reals are in essence mirror images of each other; as groups they are isomorphic.

Are addition and multiplication the only natural operations on the natural numbers whose relationship is expressed by a distributive law and a mapping like the logarithm? Are there other natural arithmetic operations hidden in the integers or reals that are related in a similar way?

For a long time algebraists have studied other ring and field structures having operations that are related to each other as addition is to multiplication. In fact, the linear, homogenous nature of ring endomorphisms suffices to endow any Abelian group with a natural ring structure. (See [5], pg. 116.)

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One group of algebraists has paid considerable attention to max-plus algebras and semirings (See [3]). These structures usually employ the operation of addition and the maximum operator: $\max\{x, y\}$. In this case, ordinary addition distributes over max. Results in this area have contributed greatly to the solution of problems in optimization and control.

The purpose of the present paper is to show that addition and multiplication are two adjacent operations in an infinite chain or graded hierarchy of binary operations on the reals and complex numbers. The chain can be extended "upwards" from multiplication and "downwards" from addition. Each operation is related to its next lower neighbor in the chain by being distributive over its neighbor. The logarithm provides the means for descending down the chain and the exponential function the means for ascending up the chain.

After we exhibit some general algebraic properties for these operations, we will proceed to investigate one of these new operations – the join operation – in considerably more detail. We shall discover that when extended to the complex numbers the join / additive structure is potentially as rich as the ordinary additive / multiplicative structure of the complex numbers.

1. The Natural Chain of Binary Arithmetic Operations

Throughout this paper $\mathbb{R}$ will denote the field of real numbers, $\mathbb{R}_0^+$ the multiplicative monoid of positive reals with 0 adjoined, and $\mathbb{R}_\infty$ the extended real number system, i.e., the real numbers with the two extreme points $\pm\infty$ adjoined and the usual operational conventions. $\mathbb{R}_\infty$ is not a field, however, since expressions like $\infty - \infty$ and ∞/∞ are not defined. Further, we will extend the definition of the natural logarithm to include 0 in its domain: $\log(0) := -\infty$. Similarly, we define $e^{-\infty} := 0$, and $e^{+\infty} := \infty$.

With these conventions we now define a countable family of binary operations on $\mathbb{R}_\infty$ as follows:

Definition 1.1. Let $x, y \in \mathbb{R}_\infty$. For each $n \in \mathbb{Z}$, we shall recursively define a binary operation $\oplus_n$ on $\mathbb{R}_\infty$ as follows. For $n = 0$, define

$$x \oplus_0 y := x + y \tag{1.1}$$

For each $n \leq 0$, define

$$x \oplus_{n-1} y := \ln(e^x \oplus_n e^y) \tag{1.2}$$

Finally, for $n \geq 0$, define

$$x \oplus_{n+1} y := \exp[\ln(x) \oplus_n \ln(y)] \tag{1.3}$$

provided $x \geq 0$ and $y \geq 0$. We shall call $\oplus_n$ an ***arithmetic operation***.

Remark 1.2. We note that for ordinary multiplication ($\times$) we have $\times = \oplus_1$. Also, we introduce the notation $\vee$ for the special case of $n = -1$: i.e., in place of $\oplus_{-1}$ we shall use $\vee$. Two operations are said to be ***adjacent*** if their associated integers are consecutive. Adjacent operations are, as we shall see below, very closely related.

Many authors, when considering only two adjacent operations, tend to use $\otimes$ for the "higher" operation and $\oplus$ for the lower one. However, since in this paper we will often consider three operations together, we will either retain the general notation of $\oplus_n$ or use $\vee$ in conjunction with $+$ and $\times$ (the latter symbol being usually omitted), depending on the context. The choice of the $\vee$ notation is motivated by its use in lattice theory for an addition-like operation. It is read as "join," "cup," or "vee" (although "union" might also be appropriate). The join terminology, which I tend to favor, is taken from lattice theory [4]. In our new algebraic system $\vee$ will play an additive role while $+$ will play a multiplicative role.

Eq. (1.2) of this definition makes sense because $e^x \geq 0$ for all $x \in \mathbb{R}_\infty$. The restriction on Eq. (1.3) prevents us from applying the logarithm to negative numbers, a restriction we shall remove when we extend the domain of definition to the complex numbers.

Remark 1.3. Eq. (1.2) appears to be a special case of an operation defined by Maslov et. al. in [2] and referenced also by Gaubert in [3]. Maslov defines: $x \oplus_h y := h\ln(e^{x/h} + e^{y/h})$. By letting $h = 1$ in Maslov's definition we obtain what I have called $\oplus_{-1}$. The semirings resulting from use of $\oplus_h$ and addition according to Maslov's definition are sometimes referred to as log-plus semirings (or semi-fields when extended with inverses, see [6]). The Maslov theory is primarily concerned with a single operation in relation to addition, parameterized by the continuous parameter h with convergence to the max operation in the limit as $h \to 0$. The theory presented here, however, is concerned with distinct operations for each n, and with the special cases of addition when $n = 0$ (by convention) and multiplication when $n = 1$.

Using the notation $f^{(n)} := \underbrace{f \circ f \circ \cdots \circ f}_{n-times}$, where $\circ$ represents function composition, it is easily verified that for $n > 0$

$$x \oplus_n y = \exp^{(n)}[\ln^{(n)}(x) + \ln^{(n)}(y)] \tag{1.4}$$

for non-negative $x, y \in \mathbb{R}_\infty$ and

$$x \oplus_{-n} y = \ln^{(n)}[\exp^{(n)}(x) + \exp^{(n)}(y)] \tag{1.5}$$

for all $x, y \in \mathbb{R}_\infty$.

2. Algebraic Properties of Arithmetic Operations

We now explore a few basic algebraic properties of adjacent operations. We will focus primarily on traversing the chain downward:

Theorem 2.1. *On $\mathbb{R}_\infty$, for all $n \in \mathbb{Z}$,*
(i) $\oplus_n$ is associative.
(ii) $\oplus_n$ is commutative.
(iii) $\oplus_n$ distributes over $\oplus_{n-1}$.

Proof. First, we note that Eq. (1.4) implies that for $x, y \geq 0$, $\ln(x \oplus_n y) = \ln(x) \oplus_{n-1} \ln(y)$. For,

$$\begin{aligned}
\ln(x \oplus_n y) &= \ln \circ \exp^{(n)}[\ln^{(n)}(x) + \ln^{(n)}(y)] \\
&= \exp^{(n-1)}[\ln^{(n)}(x) + \ln^{(n)}(y)] \\
&= \exp^{(n-1)}[\ln^{(n-1)} \circ \ln(x) + \ln^{(n-1)} \circ \ln(y)] \\
&= \ln(x) \oplus_{n-1} \ln(y),
\end{aligned}$$

(i) To prove associativity for $n < 0$, assume inductively that $\oplus_n$ is associative and consider:

$$\begin{aligned}(x \oplus_{n-1} y) \oplus_{n-1} z &= \ln\left(e^{\ln(e^x \oplus_n e^y)} \oplus_n e^z\right) \\ &= \ln\left(e^x \oplus_n e^y \oplus_n e^z\right) \\ &= \ln\left(e^x \oplus_n e^{\ln(e^y \oplus_n e^z)}\right) \\ &= x \oplus_{n-1} (y \oplus_{n-1} z).\end{aligned}$$

Similarly, for $n > 0$, and subject to the restriction $x, y \geq 0$, we have

$$\begin{aligned}(x \oplus_{n+1} y) \oplus_{n+1} z &= \exp[\ln(x \oplus_{n+1} y) \oplus_n \ln z] \\ &= \exp[\ln x \oplus_n \ln y \oplus_n \ln z] \\ &= \exp[\ln x \oplus_n \ln(y \oplus_{n+1} z)] \\ &= x \oplus_{n+1} (y \oplus_{n+1} z).\end{aligned}$$

(ii) This is an obvious consequence of the commutativity of addition.
(iii) To prove the distributive property, consider

$$\begin{aligned}x \oplus_n (y \oplus_{n-1} z) &= \exp^{(n)}[\ln^{(n)} x + \ln^{(n)}(y \oplus_{n-1} z)] \\ &= \exp^{(n)}[\ln \circ \ln^{(n-1)} x + \ln \circ \ln^{(n-1)}(y \oplus_{n-1} z)] \\ &= \exp^{(n)}[\ln(\ln^{(n-1)} x \times \ln^{(n-1)}(y \oplus_{n-1} z))] \\ &= \exp^{(n)}[\ln(\ln^{(n-1)} x \times [\ln^{(n-1)} y + \ln^{(n-1)} z])] \\ &= \exp^{(n)}[\ln(\ln^{(n-1)} x \times \ln^{(n-1)} y + \ln^{(n-1)} x \times \ln^{(n-1)} z)] \\ &= \exp^{(n)} \circ \ln[\ln^{(n-1)} x \times \ln^{(n-1)} y + \ln^{(n-1)} x \times \ln^{(n-1)} z] \\ &= \exp^{(n-1)}[\ln^{(n-1)} x \times \ln^{(n-1)} y + \ln^{(n-1)} x \times \ln^{(n-1)} z] \\ &= (x \oplus_n y) \oplus_{n-1} (x \oplus_n z).\end{aligned}$$

□

Note that $-\infty$ is an identity element for $\vee$, because

$$x \vee -\infty = \ln(e^x + e^{-\infty}) = \ln(e^x + 0) = \ln(e^x) = x.$$

It is also two-sided owing to the commutative property. Going up the chain we discover a very interesting sequence of identity elements and inverse elements. Table 2 lists some of the identities and inverses for these low order operations.

Value of n	**Common Notation**	**Identity**	**Inverse**
$n = 3$	N/A	e^e	$\exp[\exp(1/\ln\ln x)]$
$n = 2$	N/A	e	$\exp(1/\ln x)$
$n = 1$	$\times$	1	$1/x$
$n = 0$	$+$	0	$-x$
$n = -1$	$\vee$	$-\infty$	TBD

TABLE 1. Operations, Identities and Inverses

The TBD (=To Be Determined) will be cleared up in the complex case. The operation $\oplus_2$ captures the essence of exponentation. For, we note that $e^n \oplus_2 x = x^n$, and $e^m \oplus_2 e^n = e^{mn}$, for any $x \in \mathbb{R}_\infty$ and $m, n \in \mathbb{Z}$. Incidently, the commutativity of $\oplus_2$ implies the following beautiful and well-known fact:

$$x^{\ln y} = y^{\ln x}. \tag{2.1}$$

3. Algebraic Properties of the Join Operation

In this section we will begin our detailed study of the join operation $\vee$, defined for all $x, y \in \mathbb{R}_\infty$. If you think of multiplication as being "fast addition", then the $\vee$ operation can be thought of as "slow addition." Because it is the image of a sum under the logarithm mapping, it could also be called "logarithmic addition."

For positive $x \in \mathbb{N}$ we use the notation $nx = \underbrace{x + \cdots + x}_{n-times}$. Similarly, we introduce the following notation for $\vee$: $n_\vee x := \underbrace{x \vee \cdots \vee x}_{n-times}$. We also agree that $1_\vee x = x$. With this notation we note that

Lemma 3.1.

$$n_\vee x = \ln(n) + x \tag{3.1}$$

Proof. We compute:
$n_\vee x := \underbrace{x \vee \cdots \vee x}_{n-times} = \ln(e^x + \cdots + e^x) = \ln(ne^x) = \ln(n) + \ln(e^x) = \ln(n) + x.$ □

An interesting application of this result is $\underbrace{0 \vee \cdots \vee 0}_{n-times} = \ln(n)$. Also note that $\ln(nx) = n_\vee \ln(x)$. This is a consequence of the homomorphic property of the logarithm: $\ln(x+y) = \ln(x) \vee \ln(y)$. This homomorphism is in fact an isomorphism with the exp function as the inverse map. Thus we have $e^{x \vee y} = e^x + e^y$. It is also easy to verify the following two equations for integers m and n :

$$(mn)_\vee x = m_\vee(n_\vee x) \tag{3.2}$$

$$m_\vee x + n_\vee y = \ln(mn) + x + y \tag{3.3}$$

Some useful recurrence relations are the following:

$$m \vee n = [(m-1) \vee (n-1)] + 1 \tag{3.4}$$

$$(m \vee n) + 1 = (m+1) \vee (n+1) \tag{3.5}$$

One of the interesting properties of the Binomial Theorem is that it relates operations that are not adjacent but are once removed, so to speak. The ordinary result from algebra, for instance, indicates what happens when you apply integer exponentiation to a sum. We now wish to prove for our new system the analogue of this theorem by considering what happens when we apply integer multiplication to a join. For $n = 2$ we compute:

$$2(x \vee y) = 2x \vee 2_\vee(x + y) \vee 2y \tag{3.6}$$

Here we have used the following:

$$\begin{aligned}2(x \vee y) &= (x \vee y) + (x \vee y)\\&= [x + (x \vee y)] \vee [y + (x \vee y)]\\&= (x + x) \vee (x + y) \vee (y + x) \vee (y + y)\\&= 2x \vee 2_{\vee}(x + y) \vee 2y\\&= 1_{\vee}2x \vee 2_{\vee}(x_y) \vee 1_{\vee}2y,\end{aligned}$$

where we have used the convention that $1_{\vee}x = x$. A couple of useful formulas are the following:

$$n_{\vee}(x + y) + z = n_{\vee}(x + y + z) \tag{3.7}$$

$$x + y + (x \vee y) = (2x + y) \vee (x + 2y) \tag{3.8}$$

We also introduce the notation $\bigvee_{k=0}^{n} x_k = x_0 \vee \cdots \vee x_n$.

Theorem 3.2. *(Binomial Theorem)*

$$n(x \vee y) = \bigvee_{k=0}^{n} \binom{n}{k}_{\vee} [(n - k)x + ky] \tag{3.9}$$

Proof. Using the fact that $\exp(n_{\vee}x) = n\exp(x)$, and making the substitutions $u = \exp(x)$ and $v = \exp(y)$, we see that

$$\begin{aligned}\exp\{\bigvee_{k=0}^{n} \binom{n}{k}_{\vee} [(n - k)x + ky]\} &= \sum_{k=0}^{n} \binom{n}{k} \exp[(n - k)x + ky]\\&= \sum_{k=0}^{n} \binom{n}{k} \exp[(n - k)x] \exp[ky]\\&= \sum_{k=0}^{n} \binom{n}{k} u^{n-k} v^{k}\\&= (u + v)^{n}.\end{aligned}$$

Taking the logarithm of both sides yields the desired formula. □

Although we will not explore polynomials to any extent in this paper, we mention in passing that a general polynomial using addition and join can be written thus:

$$p(x) = (a_n + nx) \vee (a_{n-1} + (n - 1)x) \vee \cdots \vee (a_1 + x) \vee a_0 \tag{3.10}$$

By this time the reader should be comfortable enough with the notation to allow us to drop some of the parentheses, bearing in mind that + binds more closely than $\vee$. Therefore, we can declutter the previous equation as:

$$p(x) = a_n + nx \vee a_{n-1} + (n - 1)x \vee \cdots \vee a_1 + x \vee a_0 \tag{3.11}$$

It should be quite apparent by now that many of the ordinary arithmetic expressions and statements involving exponentiation, multiplication and addition can be readily transformed into ones involving multiplication, addition, and join, respectively. As we saw in the proof of the Binomial Theorem, many analogues of standard theorems can be successfully guessed by taking logarithms, and their proofs can be transferred back into the domain of ordinary algebra by applying the exponential function. It is this transfer principle that gives rise to so many interesting results using join and addition. It should also be borne in mind that many of the results we are demonstrating can be obtained easily for arbitrary pairs of adjacent operations. The algebra becomes a bit more complex, however, and we will leave exploration of more general cases to subsequent papers.

For now we will leave algebra behind and begin to explore the analytic properties of the join operation.

4. Analytic Properties of the Join Operation

The join operation provides a smooth approximation of the max operation. The following facts should aid in making this clear and helping us to understand the join's qualitative behavior: $\lim_{x\to-\infty}(x \vee y) = \lim_{x\to-\infty} \ln(e^x + e^y) = y$. This implies that $x \vee y \approx y$ whenever x is significantly smaller than y. On the other hand, $x \vee y \approx x$, whenever x is considerably larger than y. Thus the value of the join depends on which exponential term dominates.

The join operation, as a composite of continuous functions is itself continuous at every point in $\mathbb{R}_\infty$. Moreover, it is clearly differentiable in both arguments. We will now explore how it behaves under ordinary and partial differentiation.

4.1. **Differential Identities and Formulas.** We wish to determine how $\vee$ behaves under differentiation. If $u = u(x)$ and $v = v(x)$ are differentiable real functions of a single real variable, then the elementary formulas of the differential calculus tell us that

$$\frac{d}{dx}uv = u\frac{dv}{dx} + \frac{du}{dx}v \tag{4.1}$$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \tag{4.2}$$

An operator such as d/dx on a semiring R with these properties is called a derivation on R. We can see easily enough that d/dx is *not* a derivation on the semiring $\langle \mathbb{R}_\infty; +, 0; \vee, -\infty\rangle$. In fact, we can easily compute the general formula:

$$\frac{d}{dx}(u \vee v) = \frac{e^u\dfrac{du}{dx} + e^v\dfrac{dv}{dx}}{e^u + e^v} \tag{4.3}$$

In particular, we see that

$$\frac{d}{dx}(u \vee u) = \frac{e^u\dfrac{du}{dx} + e^u\dfrac{du}{dx}}{e^u + e^u} = \frac{2e^u}{2e^u}\frac{du}{dx} = \frac{du}{dx}. \tag{4.4}$$

If $u(x) = x$, then $\dfrac{d}{dx}(x\vee x) = 1$, from which it follows that the function $x\vee x = x+C$, which is consistent with what we already know, i.e., that $x \vee x = x + \ln(2)$, or $C = \ln(2)$.

Also note that $\dfrac{d}{dx}(x\vee -x) = \tanh(x)$, from which we derive $x\vee -x = \ln\cosh(x) + C$, to within a constant. Note that $\ln\cosh(0) = 0$ and $\ln(2) = 0 \vee 0$, which implies that $x \vee -x = \ln\cosh(x) + \ln(2) = \ln(2\cosh(x))$. This is consistent, of course, with our original definition of the join. We'll obtain even more interesting results along these lines when we extend the join to the complex numbers. Also, Eq. (4.3) will turn out to be of considerable utility in demonstrating many properties of a new derivation we will define.

We close this section with a few more formulae, the details of the proofs being simple and therefore omitted:

Theorem 4.1.

$$\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right](x \vee y) = 1 \tag{4.5}$$

$$\nabla^2 (x \vee y) = 1 - \left[\left(\frac{\partial}{\partial x}(x \vee y)\right)^2 + \left(\frac{\partial}{\partial y}(x \vee y)\right)^2\right] \tag{4.6}$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}(x \vee y) = \frac{\partial}{\partial y}\frac{\partial}{\partial x}(x \vee y) = \frac{-e^x e^y}{e^x + e^y} = -\exp\left[x + y - (x \vee y)\right] \tag{4.7}$$

$$\left[\frac{\partial}{\partial x} \vee \frac{\partial}{\partial y}\right](xy) = x \vee y \tag{4.8}$$

$$\left[\frac{\partial}{\partial x} \vee \frac{\partial}{\partial y}\right](x + y) = 1 \vee 1 = 1 + \ln 2 \tag{4.9}$$

$$\left[\frac{\partial}{\partial x} \vee \frac{\partial}{\partial y}\right](x \vee y) = \frac{e^x}{e^x + e^y} \vee \frac{e^y}{e^x + e^y} \tag{4.10}$$

Although these formulae are interesting in their own right and potentially useful in simplifying calculations and solving differential equations, what we really need is a ring (or better yet, a field) and a new derivation that parallels the classical differentiation. To accomplish this we must now turn to the complex numbers.

5. Extension of the Join to the Complex Plane

Extending the join operation to the complex plane is straightforward and has several benefits, the chief of which being that we obtain a full and rich algebraic structure that allows us to construct a new differential and integral calculus.

Before heading down that path, however, we must first take a little care in dealing with the point at infinity. In extended real analysis, we distinguish between $+\infty$ and $-\infty$. They are two different points added to the real line. In the complex plane, however, thanks to the Riemann Sphere, we usually consider only one point at infinity. This is inconvenient when we wish to have $e^{-\infty} = 0$. Although this statement makes intuitive sense in the extended real system, it is not so intuitive in the complex case. For, in the complex case all points at infinity tend to coalesce into a single point.

There are several ways to get around this issue. First, we can distinguish in terms of how the point at infinity is approached. Left and right hand limits are in some sense different. For example, if we treat an approach to ∞ with the real part becoming more and more negative, we might call this approaching ∞ from the right. Similarly, approaching ∞ from the left would mean that the real part is becoming more and more positive. This would be consistent with $\lim_{z\to-\infty} e^z = 0$ and $\lim_{z\to\infty} e^z = \infty$. Obviously, since these two values are not the same, if we treat $+\infty$ and $-\infty$ as the same point at infinity, then the exponential function has a discontinuity at that point. However, if we keep these points distinct, we need not worry about the limits being different.

Another approach would be to adjoin four new points to $\mathbb{C}$: The real infinities $+\infty$ and $-\infty$ and the pure imaginary infinities $+i\infty$ and $-i\infty$. The algebraic operations using these symbols will make sense as long as we don't try to cancel infinities in undefined ways. This approach destroys, however, the field structure of the complex numbers.

For our purposes it is not necessary to proceed in this manner. We can retain a field structure on $\mathbb{C}$ by adjoining a single new infinite quantity: $-\infty$. Thus $\langle \mathbb{C} \cup -\infty; \vee, -\infty; +, 0\rangle$ becomes a field with the conventions that $-\infty + z = -\infty$ and $-\infty \vee z = z$ for all $z \in \mathbb{C}$. Addition is the "multiplicative" operation of the field and join is the "additive" operation.

To complete the field structure we need to show that the join operation in the complex plan affords an "additive" inverse. However, before we do this, we need to discuss the multifunction nature of the complex logarithm.

Since the complex logarithm is a multifunction that takes on infinitely many values for each point of the complex plane, we need to take care when asserting equality of expressions involving logarithms. Because the principal branch of the logarithm $\mathrm{Log}(z)$ is not a homomorphism (i.e., it does not always map products into sums), we will avoid limiting ourselves to this branch. Therefore, we shall use the general complex logarithm $\log(z) = \ln|z| + i\arg(z)$ and be deliberately vague about which branch of the logarithm we restrict ourselves to in any particular situation. In many cases our equations will, strictly speaking, be equivalences modulo $2\pi i$. For further discussion of various branches of the logarithm, see Ahlfors [1].

We can now define our complex join as:

$$z_1 \vee z_2 := \log\left[\exp(z_1) + \exp(z_2)\right] \tag{5.1}$$

By the convention we have adopted regarding the absorptive nature of $-\infty$, we can see that $\log(z) \to -\infty$ continuously as $z \to 0$.

The inverse of z with respect to the join operation will sometimes be denoted by $\tilde{z}$. However, we can readily compute that

$$\tilde{z} = z + i\pi \tag{5.2}$$

For,

$$\begin{aligned} z \vee (z + i\pi) &= \log(e^z + e^{z+i\pi}) \\ &= \log(e^z + e^z e^{i\pi}) \\ &= \log(e^z - e^z) \\ &= \log(0) \\ &= -\infty. \end{aligned}$$

We also notice that $\log(-z) = \log(z) + i\pi$, for

$$\begin{aligned} \log(-z) \vee \log(z) &= \log(e^{\log(-z)} + e^{\log(z)}) \\ &= \log(-z + z) \\ &= \log(0) \\ &= -\infty. \end{aligned}$$

With the definition of the inverse in hand, we are now in a position to define a new kind of derivative:

Definition 5.1. Let $G \subseteq \mathbb{C}$ be an open subset of $\mathbb{C}$, $f : G \to \mathbb{C} \cup \{-\infty\}$, and suppose that for some $z \in \mathbb{C} \cup \{-\infty\}$ the limit

$$\lim_{h \to -\infty} \{[f(z \vee h) \vee (f(z) + i\pi)] - h\} \tag{5.3}$$

exists and is finite. Then we shall say that f is $\vee$-***differentiable*** at z and denote the limit by $D_\vee f(z)$, called the $\vee$-***derivative*** of f at z.

Remark 5.2. Whenever there is little danger of confusion and the meaning is clear from the context, we shall simply write $Df(z)$ instead of $D_\vee f(z)$. Later, when we once again consider more general operations on $\mathbb{C}$ and define more generalized derivatives, we will use the notation D_{-1} instead of either D or $D_\vee$.

Remark 5.3. This definition results from a direct translation of the ordinary difference quotient $\dfrac{f(z+h) - f(z)}{h}$ using the previously mentioned transfer principle that sends sums into joins, subtractions into joins of "additive" inverses, and quotients into subtraction. Thus this definition is the natural analogue of the ordinary derivative.

Some basic properties of $\vee$-differentiation that are near obvious analogues of their classical counterparts are the following:

Theorem 5.4. *Let n be a non-negative integer, a a constant complex number, z an indeterminate complex variable, and both f and g complex functions analytic and $\vee$-differentiable on an open set in $\mathbb{C}$. Then*

(i) $D(a) = -\infty$
(ii) $D(z) = 0$
(iii) $D(a + z) = a$
(iv) $D(n_\vee f) = n_\vee D(f)$
(v) $D(f \vee g) = D(f) \vee D(g)$
(vi) $D(f + g) = [f + D(g)] \vee [D(f) + g]$
(vii) $D(nz) = n_\vee (n-1)z = (n-1)z + \ln(n)$
(viii)) $D(z^n) = \log\left[\dfrac{nz^{n-1}e^{z^n}}{e^z}\right] = z^n - z + (n-1)\log(z) + \ln(n)$
(ix) $D^n(nz) = \ln 1 + \cdots + \ln n$

Proof. Although these proofs are elementary, they serve to illustrate some important techniques that are peculiar to reasoning with joins. Later, we shall see that each of these formulae can be simply derived from a single principle expressing the relationship between $\vee$-differentiation and ordinary differentiation.

(i) Observe that

$$\begin{aligned} D(a) &= \lim_{h \to -\infty} \{[a \vee (a + i\pi)] - h\} \\ &= \lim_{h \to -\infty} \{-\infty - h\} \\ &= \lim_{h \to -\infty} \{-\infty\} \\ &= -\infty. \end{aligned}$$

(ii) Again, we compute

$$\begin{aligned}
D(z) &= \lim_{h\to-\infty}\{[z \vee h \vee (z+i\pi)] - h\} \\
&= \lim_{h\to-\infty}\{(z-h) \vee (h-h) \vee (z+i\pi-h)\} \\
&= \lim_{h\to-\infty}\{(z-h) \vee 0 \vee (z+i\pi-h)\} \\
&= \lim_{h\to-\infty}\{(z-h) \vee (z-h+i\pi) \vee 0\} \\
&= \lim_{h\to-\infty}\{-\infty \vee 0\} \\
&= \lim_{h\to-\infty}\{0\} \\
&= 0.
\end{aligned}$$

(iii) Similarly

$$\begin{aligned}
D(a+z) &= \lim_{h\to-\infty}\{[a+(z \vee h) \vee (a+z+i\pi)] - h\} \\
&= \lim_{h\to-\infty}\{[(a+z) \vee (a+h) \vee (a+z+i\pi)] - h\} \\
&= \lim_{h\to-\infty}\{(a+z-h) \vee (a+h-h) \vee (a+z+i\pi-h)\} \\
&= \lim_{h\to-\infty}\{(a+z-h) \vee (a+z-h+i\pi) \vee a\} \\
&= \lim_{h\to-\infty}\{-\infty \vee a\} \\
&= \lim_{h\to-\infty}\{a\} \\
&= a.
\end{aligned}$$

(iv) First note that

$$\begin{aligned}
(f \vee g)(z \vee h) \vee ([(f \vee g)(z)] + i\pi) &= f(z \vee h) \vee g(z \vee h) \vee ([f(z) \vee g(z)] + i\pi) \\
&= f(z \vee h) \vee g(z \vee h) \vee (f(z)+i\pi) \vee (g(z)+i\pi) \\
&= f(z \vee h) \vee (f(z)+i\pi) \vee g(z \vee h) \vee (g(z)+i\pi)
\end{aligned}$$

Next, distribute the $-h$ over the joins and take the limit as h approaches $-\infty$. The result then follows.

(v) We compute using induction and the previous result

$$\begin{aligned}
D(n_\vee f) &= D(f \vee \cdots \vee f) \\
&= n_\vee D(f).
\end{aligned}$$

(vi) First consider the "numerator" of the "difference quotient":

$$\begin{aligned}
[f(z \vee h) + g(z \vee h)] \vee [(f(z)+g(z)) + i\pi] &= [f(z \vee h) + g(z \vee h)] \vee [f(z) + g(z) + i\pi] \\
&= [f(z \vee h) + g(z \vee h)] \vee [f(z \vee h) + g(z)] \\
&\quad \vee [f(z \vee h) + g(z) + i\pi] \vee [f(z) + g(z) + i\pi] \\
&= (f(z \vee h) + [g(z \vee h) \vee (g(z) + i\pi)]) \\
&\quad \vee (g(z) + [f(z \vee h) \vee (f(z) + i\pi)])
\end{aligned}$$

The result then follows upon distributing $-h$ and taking limits.

(vii) To show that $D(nz) = (n-1)z + \ln n$, use (vi).

(viii) To prove $D(z^n) = \log\left(\dfrac{nz^{n-1}e^{z^n}}{e^z}\right)$, consider the "difference quotient" $[(z \vee h)^n - h] \vee [z^n + i\pi - h]$, in which we have already distributed the $-h$. This becomes

$$\begin{aligned}\log\{\exp[(z \vee h)^n - h] + \exp[z^n + i\pi - h]\} &= \log\left\{\frac{\exp[(z \vee h)^n] + \exp[z^n + i\pi]}{\exp(h)}\right\}\\ &= \log\left\{\frac{\exp[(z \vee h)^n] - \exp[z^n]}{\exp(h)}\right\}\end{aligned}$$

Now we can apply l'Hospital's Rule, since the numerator and the denominator each approach 0 as $h \to -\infty$. Thus, we let $f(h) = \exp[(z \vee h)^n - h] + \exp[z^n + i\pi - h]$ and $g(h) = \exp(h)$. Then, computing ordinary derivatives with respect to h of f and g we obtain:

$$\begin{aligned}f'(h) &= \exp[(z \vee h)^n] \cdot n \cdot (z \vee h)^{n-1} \cdot \frac{d}{dh}(z \vee h)\\ &= \exp[(z \vee h)^n] \cdot n \cdot (z \vee h)^{n-1} \cdot \frac{\exp(h)}{\exp(z) + \exp(h)}\end{aligned}$$

by Eq. (4.3). Further, $g'(h) = \exp(h)$. Therefore,

$$\begin{aligned}\frac{f'(h)}{g'(h)} &= \frac{\exp[(z \vee h)^n] \cdot n \cdot (z \vee h)^{n-1} \cdot \dfrac{\exp(h)}{\exp(z) + \exp(h)}}{\exp(h)}\\ &= \frac{\exp[(z \vee h)^n] \cdot n \cdot (z \vee h)^{n-1}}{\exp(z) + \exp(h)}\end{aligned}$$

Taking the limit as $h \to -\infty$, we have $(z \vee h)^n \to (z \vee -\infty)^n) = z^n$ and thus,

$$\begin{aligned}\frac{f'(h)}{g'(h)} &= \frac{\exp[(z \vee h)^n] \cdot n \cdot (z \vee h)^{n-1}}{\exp(z) + \exp(h)}\\ &\to \frac{\exp[z^n] \cdot n \cdot z^{n-1}}{\exp(z)}\\ &= \frac{nz^{n-1}e^{z^n}}{e^z}.\end{aligned}$$

By l'Hospital's Rule $\dfrac{f(h)}{g(h)}$ approaches the same limit as $\dfrac{f'(h)}{g'(h)}$ and, by the continuity of the logarithm, we obtain

$$D(z^n) = \log\left[\frac{nz^{n-1}e^{z^n}}{e^z}\right] = z^n - z + (n-1)\log(z) + \ln(n).$$

(ix) Finally, we compute

$$\begin{aligned}
D^n(nz) &= D^{n-1}D(nz)\\
&= D^{n-1}n_\vee(n-1)z\\
&= n_\vee D^{n-1}(n-1)z\\
&= n_\vee(n-1)_\vee D^{n-2}(n-2)z\\
&\vdots\\
&= n_\vee(n-1)_\vee\cdots_\vee(n-n+1)_\vee(n-n)z\\
&= (n!)_\vee 0\\
&= \ln(n!)\\
&= \ln 1+\cdots+\ln n
\end{aligned}$$

□

Remark 5.5. The use of l'Hospital's Rule in the demonstration in part (viii) of Theorem 5.4 is not restricted to that specific instance. For a general analytic function f we consider the following:

$$\begin{aligned}
Df(z) &= \lim_{h\to-\infty}\{[f(z\vee h)\vee(f(z)+i\pi)]-h\}\\
&= \lim_{h\to-\infty}\{\log[\exp[f(z\vee h)]+\exp(f(z)+i\pi)]-\log[\exp(h)]\}\\
&= \lim_{h\to-\infty}\log\left\{\frac{\exp[f(z\vee h)]+\exp[f(z)+i\pi]}{\exp(h)}\right\}
\end{aligned}$$

Since the limit of the numerator and the denominator each approach 0, we can apply l'Hospital's Rule and take the (ordinary) derivative of each with respect to h:

$$\begin{aligned}
Df(z) &= \lim_{h\to-\infty}\log\left\{\frac{\exp[f(z\vee h)]+\exp[f(z)+i\pi]}{\exp(h)}\right\}\\
&= \lim_{h\to-\infty}\log\left\{\frac{\exp[f(z\vee h)]\cdot f'(z\vee h)\cdot\dfrac{\exp(h)}{\exp(z)+\exp(h)}}{\exp(h)}\right\}\\
&= \lim_{h\to-\infty}\log\left\{\frac{\exp[f(z\vee h)]\cdot f'(z\vee h)}{\exp(z)+\exp(h)}\right\}\\
&= \log\left\{\frac{\exp[f(z)]f'(z)}{\exp(z)}\right\}\\
&= f(z)+\log[f'(z)]-z.
\end{aligned}$$

In other words, we have proved:

Theorem 5.6. *If $f: G\to\mathbb{C}$ is an analytic and $\vee$-differentiable function defined on a open subset $G\subseteq\mathbb{C}$, then for all $z\in\mathbb{C}$,*

$$Df(z)=f(z)+\log[f'(z)]-z. \tag{5.4}$$

Remark 5.7. This general fact supplies some more elegant proofs of the special formulae in Theorem 5.4. For example, the following is a proof of the "product rule" for $\vee$-differentiation contained in (iv) of that theorem:

$$\begin{aligned} D(f+g)(z) &= f(z)+g(z)+\log\left[f'(z)+g'(z)\right]-z \\ &= f(z)+g(z)+(\log[f'(z)]\vee\log[g'(z)])-z \\ &= [f(z)+g(z)+\log[f'(z)]-z]\vee[f(z)+g(z)+\log[g'(z)]-z] \\ &= [Df(z)+g(z)]\vee[f(z)+Dg(z)]\,. \end{aligned}$$

Remark 5.8. Eq (5.4) also enables us to prove new results such as

(i) $D[ae^{bz}] = ae^{bz}+(b-1)z+\log(ab)$
(ii) $D[\exp(z)] = \exp(z)$
(iii) $D[\log(z)] = -z$

It is interesting that the exponential function retains with respect to the $\vee$-derivative the invariance it enjoys with respect to the ordinary derivative. Additionally, the $\vee$-derivative of logarithm takes on the analogue value: $-z$ is to $+$ what $1/z$ is to $\times$.

There are many more formulae analogous to classical ones that we could prove, the chain rule being one of the more important ones. However, space limitations prevent us from presenting them here.

6. Generalizations of the Derivative

The approach used to define the $\vee$-derivative can be taken to define a generalized derivative for any n. Let us introduce the following new notation: For each $n\in\mathbb{Z}$, let $\sim_n z$ denote the inverse of $z\in\mathbb{C}$ with respect to the operation $\oplus_n$, and let 0_n denote the identity element with respect to $\oplus_n$. Thus $0_{-1}=-\infty$, $0_0=0$ and and $0_1=1$. Also, $\sim_{-1} z = z+i\pi$, $\sim_0 z=-z$, and $\sim_1 z = 1/z$. Recalling that $\vee=\oplus_{-1}$, and introducing the notation D_{-1} for our new derivative, we see that

$$\begin{aligned} D_{-1}f(z) &= \lim_{h\to-\infty}\{[f(z\vee h)\vee(f(z)+i\pi)]-h\} \\ &= \lim_{h\to 0_{-1}}\{[f(z\oplus_{-1}h)\oplus_{-1}(\sim_{-1}f(z))]\oplus_0[\sim_0 h]\} \end{aligned}$$

Formally, this suggests that we attempt to generalize the derivative as follows:

Definition 6.1. For any $n\in\mathbb{Z}$, $f:G\to\mathbb{C}$ analytic on an open subset $G\subseteq\mathbb{C}$, and $z\in G$, define the ***n-derivative*** of f at the point z as

$$D_n f(z) := \lim_{h\to 0_n}\{[f(z\oplus_n h)\oplus_n(\sim_n f(z))]\oplus_{n+1}[\sim_{n+1} h]\} \tag{6.1}$$

Remark 6.2. By applying this to the case of $n = 1$, and recalling both that $x \oplus_2 y = \exp[\log(x)\log(y)]$ and $\sim_2 z = \exp\left[\frac{1}{\log(z)}\right]$, we compute:

$$\begin{aligned}
D_1 f(z) &= \lim_{h\to 1} \exp\left\{\log\left[\frac{f(zh)}{f(z)}\right]\log\left[\exp\left(\frac{1}{\log(h)}\right)\right]\right\} \\
&= \lim_{h\to 1} \exp\left\{\frac{\log\left[\frac{f(zh)}{f(z)}\right]}{\log(h)}\right\} \\
&= \lim_{h\to 1} \exp\left\{\frac{\log[f(zh)] - \log[f(z)]}{\log(h)}\right\}
\end{aligned}$$

which is clearly reminiscent of a difference quotient. We can refine this further with the aid of l'Hospital's Rule as follows:

$$\begin{aligned}
D_1 f(z) &= \lim_{h\to 1} \exp\left\{\frac{\log[f(zh)] - \log[f(z)]}{\log(h)}\right\} \\
&= \lim_{h\to 1} \exp\left[\frac{f'(zh)zh}{f(zh)}\right] \\
&= \exp\left[\frac{f'(z)z}{f(z)}\right].
\end{aligned}$$

If this formula makes sense, then we should also be able to generalize Eq. (5.4) in the following manner:

$$D_n f(z) = f(z) \oplus_{n+1} \log\left[D_{n+1} f(z)\right] \oplus_{n+1} \left[\sim_{n+1} z\right] \tag{6.2}$$

This in turn should allow us to compute the ordinary 0-derivative D_0 in terms of the next higher 1-derivative D_1. In other words,

$$\begin{aligned}
f'(z) = D_0 f(z) &= f(z) \oplus_1 \log\left[D_1 f(z)\right] \oplus_1 \left[\sim_1 z\right] \\
&= f(z)\cdot\log\left[D_1 f(z)\right]\cdot\frac{1}{z} \\
&= f(z)\cdot\log\exp\left[\frac{f'(z)z}{f(z)}\right]\cdot\frac{1}{z} \\
&= f(z)\cdot\frac{f'(z)z}{f(z)}\cdot\frac{1}{z} \\
&= f'(z).
\end{aligned}$$

We note that the exponential function is again invariant with respect to D_1, just as it is with respect to D_{-1} and $D_0 = \frac{d}{dz}$. For,

$$\begin{aligned}
D_1 \exp(z) &= \exp\left[\frac{\frac{d}{dz}\exp(z)\cdot z}{\exp(z)}\right] \\
&= \exp\left[\frac{\exp(z)\cdot z}{\exp(z)}\right] \\
&= \exp\left[z\right]
\end{aligned}$$

We are tempted to conjecture that the exponential function is probably invariant with respect to every D_n as defined in Eq. (6.2). However, we choose to leave this for subsequent publications.

Naturally, these results are primarily formal at this point and need to be put on more solid analytical footing by careful consideration of domains of definition and branches of the logarithm. However, their formal consistency is very encouraging and seems to indicate that the generalized derivative, along with the generalized operations, yields a countably infinite hierarchy of structures on which to conduct complex analysis.

7. Conclusions and Prospects for Further Research

[redacted]

It should be apparent that we have in this theory the beginnings of a potentially fruitful field of study. A number of the results are sufficiently compact and appealing to warrant further exploration in this area.

It is important to realize that the each operation in the chain is fully compatible with the existing topological and algebraic structure of the field of complex numbers. This is due to the continuous and smooth nature of the logarithm and exponential functions. Because these new operations can be related directly to the classical operations of addition and multiplication, they present not merely an alternative representation of the legacy field structure but rather an enrichment of it. It is this key fact that seems to hold out so much promise for further results and new insights into old problems. It also seems to indicate that we can break out of the two-operation mold of traditional ring theory and begin to explore more intricate structures with many more intimately related operations.

In subsequent investigations I will focus on exploring in more depth the relationship between the new derivative and the classical one and on defining integration. In particular, I will show that the generalized derivative as given by Eq. (6.2) is indeed a ring derivation for each n. I will also attempt to gain more insight into the geometric interpretation of the join and the $\vee$-derivative in the spirit of Needham[7]. Furthermore, I will attempt to develop a systematic theory of infinite joins analogous to that of infinite series and to exhibit some new infinite join representations of various important functions and constants.

The relationship between these natural operations and the max operation and max-plus rings and algebras in general would also seem to be of interest. It is easy to prove, for example, that $\max\{x, y\} \leq x \oplus_n y$ for every $n \in \mathbb{Z}$. It appears likely that $\max\{x, y\} = \lim_{n \to -\infty} x \oplus_n y$. However, whereas max is not a smooth operation, $\oplus_n$ is. This might be of benefit in applications to nonlinear systems theory where differentiability is always a desirable quality and the potential for transforming non-linear problems into linear ones is always attractive.

The territory opened up by this research topic appears to be potentially vast and certainly much larger than can be adequately explored by a single researcher. It is my hope that this paper will spark the interest of other researchers and that the number of new results and applications of this field will grow rapidly.

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