One way to understand $\lim_{n \to \infty} (1 + \frac{1}{n})^n$

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About e

$e = 2.718281828\cdots$ is a mathematical constant called **Euler's number** after the Swiss mathematician Leonhard Euler.

Definitions of e

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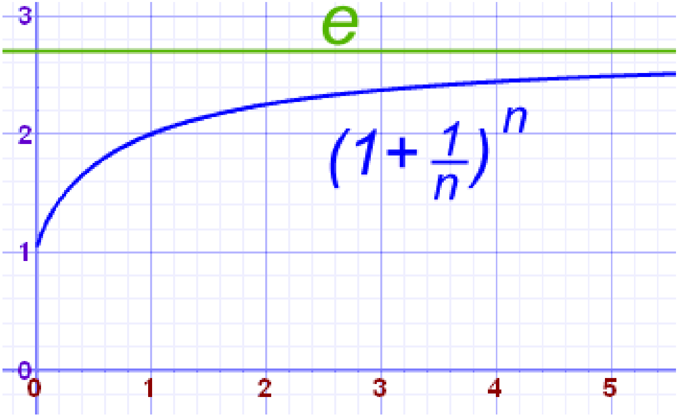
$$e \equiv \sum_{n=0}^{\infty} \frac{1}{n!}$$

Visualization

Take a look at the following graph



Question

Why is e defined as $\lim_{n\to\infty}(1+\frac{1}{n})^n$, not anything else?

Recall

Let's first recall the definition of the derivative:

Definition

The **derivative** of $f(x)$ with respect to x is the function $f'(x)$ and is defined as,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Ideas

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$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h} \\ &= a^x\left(\lim_{h\to 0} \frac{a^h - 1}{h}\right) \end{aligned}$$

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Notice here that the derivative of $f(x)$ is equal to a multiple of itself.

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- So for small values of h, we can write:

$$a^h - 1 \approx h \implies a^h \approx 1 + h \implies a \approx (1 + h)^{\frac{1}{h}}.$$

We are almost there

If we replace h by $\frac{1}{n}$, then

$$a \approx (1 + \frac{1}{n})^n.$$

Finally

The approximation gets better as n gets larger, then

$$a = \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

so that $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$

Finally

The approximation gets better as n gets larger, then

$$a = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

so that $\lim_{h \to 0} \frac{a^h - 1}{h} = 1$ (proportional constant).

Recall

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Recall

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$
$$= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$
$$= a^x \left(\lim_{h \to 0} \frac{a^h - 1}{h} \right)$$

Thus,

$$a = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \implies f'(x) = a^x \cdot 1 = a^x.$$

e

Why not give a a new name since it is a constant? Call it e!

$$e \equiv \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Thanks for listening!

Questions?