RESEARCH STATEMENT

1. Introduction

I work with non-associative algebras, Lie groups, and Lie algebras. In particular, I study the structure of a non-compact real form of E_6, using an explicit octonionic construction of its associated exceptional Lie group. This construction provides insight into triality and supersymmetry properties of the group. Additionally, it produces towers of Lie subalgebras with inclusions not usually suggested by the classical study of root and Dynkin diagrams. This knowledge of the arrangement of subgroups of E_6 is essential to attempts to unify the four fundamental forces of nature, which are related to the Lie groups $SU(3)$, $SU(2)$, $U(1)$, and the Lorentz group $SO(3,1)$.

Properties of Lie matrix groups are usually studied through their associated Lie algebras. Simple real Lie algebras are classified by complexifying the base field and constructing its root and Dynkin diagram, from which it is clear the algebra belongs to one of the four infinite families of algebras A_n, B_n, C_n, or D_n, or to one of five exceptional cases: G_2, F_4, E_6, E_7, or E_8. While the infinite families correspond to special unitary or orthogonal matrix groups over reals, complex numbers, or quaternions, the exceptional algebras are related to the octonions, the largest of the four normed division algebras.

The octonions $\mathbb{O}$ are an eight dimensional vector space over $\mathbb{R}$, containing seven orthogonal square roots of -1. Their multiplication is not associative. However, because any two octonions determine a quaternionic associative subalgebra, the octonions are alternative, or almost associative. Both the quaternions and the octonions are accessible to undergraduate students and standard linear algebra problems provide an unlimited supply of good student projects.

In 1993, Manogue and Schray constructed explicit octonionic matrix transformations which utilized alternativity [1]. Using octonionic spinors $\psi \in \mathbb{O} \times \mathbb{O}$ and vectors V, which are 2×2 hermitian octonionic matrices, they used nesting - applying one basic transformation after another - to produce well-defined transformations. Their construction produced a chain of subgroups

$$G_2 \subset SO(7) \subset SO(8) \subset SO(9) \subset SO(9,1)$$

where the smallest group, G_2, is the group of automorphisms of the octonions. My own work generalizes their construction to the 3×3 case, and produces constructions for E_6 and F_4.

2. Results

E_6 is the exceptional Lie group preserving $det(X)$, where X is a 3×3 octonionic hermitian matrix. This group exhibits triality, which is a symmetry between vectors, spinors, and dual spinors. Because it is 3×3 hermitian, there are three natural ways to identify vectors V, spinors ψ, and conjugate spinors $\psi^\dagger$ in X. These identifications, referred to as types I, II, and III,

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provide three copies of $SO(9,1)$ in E_6 (see **Fig. 1**). As the dimension of $SO(9,1)$ is 45 but the dimension of E_6 is 78, there are many dependencies among the three copies of $SO(9,1)$ in E_6.

To deal with the size of the group and the non-associativity of the octonions, I wrote Maple and Perl programs to compute the product in the algebra. These programs determined dependencies among the algebra elements, thereby suggesting dependencies among the group generators. Additionally, given a list of algebra elements, they produced the product table and used color to indicate whether subalgebras closed. These programs helped produce the following results:

(1) Triality: Strong version
There is only one G_2, as the type I, II, and III versions of any G_2 transformation are identical. Equivalently, each G_2 transformation affects the vector, spinor, and dual spinor in the same way.

(2) Triality: Weak version
The type I $SO(8)$ contains the type II and type III $SO(8)$ groups. This triality is clearly indicated by the Dynkin diagram of $SO(8)$. Further, any two of the three $SO(7)$ groups generate the $SO(8)$.

(3) F_4, the 52-dimensional exceptional Lie group preserving $Tr(X)$, consists of $SO(8)$ and its three $SO(9)$ extensions.

(4) $SO(3,1)$ can be independent of $SO(6)$. Due to triality, $SO(3,1)$ is independent of $SU(3)$.

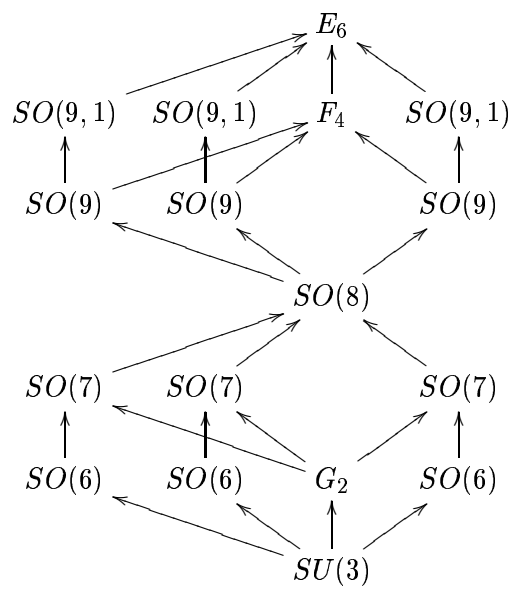
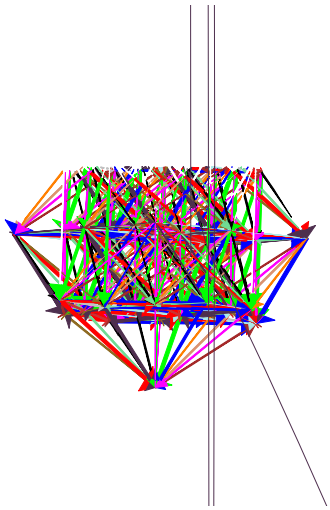


Figure 1. Triality's effect on subgroups of E_6

Root and Dynkin diagrams suggest $SO(8)$ has triality, but they can not explain how the subgroups of E_6 are arranged with respect to each other. This is particularly important with regard to the Lorentz group and $SU(3)$. Clearly, the type I $SO(3,1)$ group is independent of the type I $SO(6)$ group. But each $SO(6)$ contains $SU(3)$, as does G_2. Because of the triality exhibited by G_2, these $SU(3)$ are in fact identical, and $SU(3)$ is thus independent of the three choices of $SO(3,1)$. The specific knowledge of how the groups are arranged inside E_6 is important for finding the right subgroups $SU(3) \times SU(2) \times U(1)$ and $SO(3,1)$ needed for a unified field theory.

Subalgebras usually correspond to subdiagrams of root diagrams. These diagrams provide an equivalent geometric description of the algebra. I also wrote a program to draw each algebra's root diagram, including the 4 and 6 dimensional diagrams of F_4 and E_6, and project them into two or three dimensions. The root diagram of F_4 is shown in **Figure 2**[1]. However, root diagrams require complexified algebras. This noncompact real form of E_6 contains a subgroup structure, indicated in **Figure 3**, that is not based on root diagrams[2][3].

[1]An interactive version of this projected diagram, which can be rotated in $\mathbb{R}^3$, is at http://oregonstate.edu/~wangbera. That page contains a link to another version, where the projection direction smoothly changes through a plane in $\mathbb{R}^4$.