Quaternions, Octonions, and Student Projects

The octonions are an 8-dimensional normed division algebra which is, unfortunately, neither associative nor commutative. They contain the quaternions, a 4 dimensional normed division algebra which is associative but is still not commutative. Just as the complexes are the result of adding *i*,a square root of -1, to the reals, the quaternions are the result of adding *j*, another square root of -1, to the complexes. In the case of the quaternions though, we must also add the product of *i* and *j,* which is called *k* and also is a square root of -1. To get the octonions, we again add *l,* another square root of -1, and the product of *i, j,* and *k* with *l* to get *il, jl,* and *kl.*

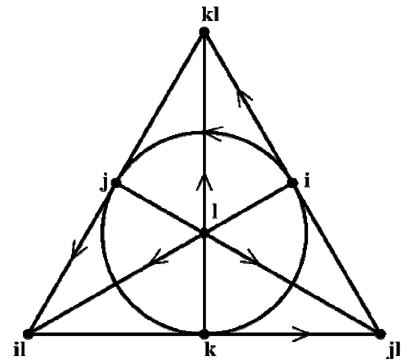

Figure 1: Octonion Multiplication Table, containing 7 “lines”

The octonion multiplication table is indicated in **Figure 1**, and it contains the quaternion multiplication table which is represented by the central circle. In each case, multiplying two elements on one “line” (including the circular “line” in the center) gives the line’s third element, with the relative sign determined by agreement with the line’s arrow. For instance, $i \cdot j = k$ but $j \cdot i = -k$. The multiplication of the octonions is non-associative, as $i \cdot (j{\cdot}l) = i \cdot (jl) = -kl$ but $(i \cdot j) \cdot l = (k) \cdot l = kl$. Each “line” in **Figure 1** is a quaternionic subalgebra. Because any two octonions lie in a quaternionic subalgebra, multiplication involving only those two octonions and their product is associative. This makes the octonions alternative.

Quaternions and octonions put a new twist on typical linear algebra problems. They are great examples of non-matrix objects which are non-commutative (and non-associative). Linear equations pose interesting problems. Because they are normed division algebras, students can use the fact that multiplicative inverses of non-zero elements are unique. Yet, the commutativity and associativity issues produce complications. Students must grapple with the realization that solutions will depend upon which algebra they are working with. For instance, when solving $axa=b$, $a(xb)=c,$ or $(ax)b=c$, the solution will depend upon whether *a, b, c,* and *x* are all part of the same complex or quaternionic subalgebra, or if they are indeed in the full octonionic algebra. Once students have formulated solutions to the above problems (given explicit parameter values of *a, b,* and *c*), they can generalize their solutions to arbitrary parameters as well as more general linear equations, for instance $a((xb)c) + (dx)f = g.$

The octonions are very geometric. Conjugating an octonion *x* first by an octonion *a* and then by *b*, that is forming $b(axa^{-1})b^{-1}$, has interesting geometric results. The octonion *x* can be decomposed into four parts: Two parts parallel to *a* and *b*, a third part parallel to the product of *a* and *b*, and the final part perpendicular to the quaternionic algebra spanned by *a, b,* and *c.* The double conjugation will produce a rotation in the plane of *x* containing the *a* and *b* pieces, but will not affect the other two pieces. Of course, this conjugation will have different effects (i.e. reflections) if either *a* or *b* is real, or if *a* is a real multiple of *b.* This calculation has both geometric and algebraic concepts which challenge student understanding of rotations and reflections. These projects lend themselves to discussions of Lie groups as well as using computer algebra and graphing systems.

A third project involves finding eigenvalues and eigenvectors of n x n Hermitian octonionic matrices. Students know that such complex matrices have *n* real eigenvalues, that eigenvectors corresponding to different eigenvalues will be orthogonal, and that there is an orthonormal basis of C^n of eigenvectors for each of those matrices. These results have analogues for the quaternionic or octonionic matrices, and are good projects. Over the larger division algebras, the eigenvalues of such matrices are not necessarily real, and multiples of eigenvectors do not necessarily remain eigenvectors. In the quaternionic case, the three statements above can be generalized for right eigenvalues (as opposed to left eigenvalues), and advanced students understanding those results can attempt the octonionic results.

These octonion projects can be simplified or made more difficult by looking at specific or generalized cases, and are thus appropriate for students at a variety of levels. In addition, quaternions and octonions also have physical applications. One projects is related to the Pauli spin matrices, which may interest physics students. Quaternions are used in computer graphics to produce rotations in three dimensions, and there are projects which would be of interest to computer science majors as well. In any case, I look forward to working with students on projects which are of an interest to both of us.