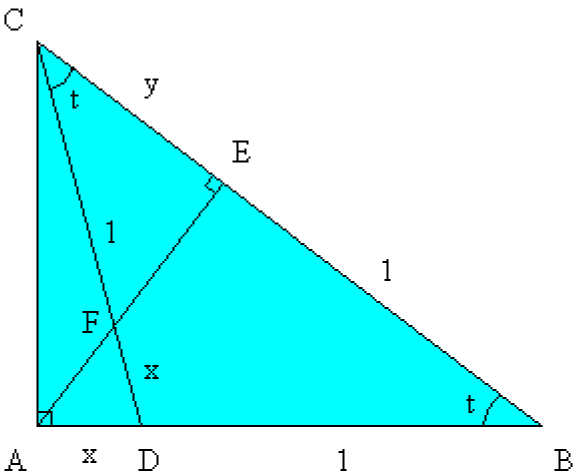
Geometric Solution

Let AD = x, CE = y, and $\angle$ ABC = t. Let AE and CD meet at F.
Since $\triangle$ BCD is isosceles, $\angle$ BCD = t.
Hence $\angle$ CFE = 90° − t, and so $\angle$ DFA= 90° − t.
Since also $\angle$ FAD = $\angle$ EAB = 90° − t, $\triangle$ DFA is isosceles, and so DF = AD = x.
Hence CF = 1 − x.



Triangles ABE and CFE are similar, as each contains a right angle, and $\angle$ ABC = $\angle$ ECF.
Hence y/(1 − x) = 1/(1 + x), and so
y = (1 − x)/(1 + x)(1)

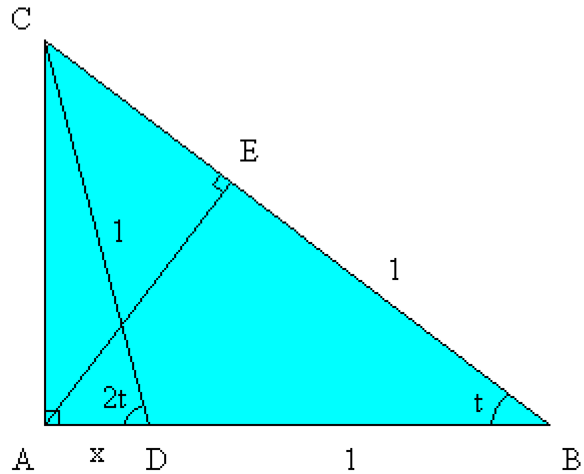
Triangles ABC and ABE are similar, as each contains a right angle, and $\angle$ ABC = $\angle$ ABE.
Hence (1 + x)/(1 + y) = 1/(1 + x), and so $(1 + x)^2 = 1 + y$.

Substituting for y from (1), we obtain
$(1 + x)^2 = 1 + (1 - x)/(1 + x) = 2/(1 + x)$.
Hence $(1 + x)^3 = 2$.

Therefore the length of AD is $\sqrt[3]{2} - 1$.

Trigonometric Solution

Let AD = x, and $\angle$ ABC = t.
Since $\triangle$ BCD is isosceles, $\angle$ BCD = t.
We also have $\angle$ BCA = 90° − t, and so $\angle$ DCA = 90° − 2t.
Hence $\angle$ ADC = 2t.



Considering triangles ABE and ADC, we obtain, respectively
cos t = 1/(1 + x)
cos 2t = x

Applying double-angle formula $\cos 2t = 2\cos^2 t - 1$, we get
$x = 2/(1 + x)^2 - 1$

Hence $(1 + x) = 2/(1 + x)^2$, from which $(1 + x)^3 = 2$.

Therefore the length of AD is $\sqrt[3]{2} - 1$.

Solve the equation $\sqrt{4+\sqrt{4-\sqrt{4+\sqrt{4-x}}}} = x$.

(All square roots are to be taken as positive.)

Consider $f(x) = \sqrt{4+\sqrt{4-x}}$.
Then $f(f(x)) = \sqrt{4+\sqrt{4-\sqrt{4+\sqrt{4-x}}}} = x$.
A solution to f(x) = x, if it exists, will also be a solution to f(f(x)) = x.

Solving f(x) = x

Consider, then, $f(x) = \sqrt{4+\sqrt{4-x}} = x$.

Let $y = \sqrt{4-x}$. Then $y^2 = 4 - x$.
We also have $x = \sqrt{4+y}$, from which $x^2 = 4 + y$.

Subtracting, we have $x^2 - y^2 = x + y$.
Hence $(x + y)(x - y - 1) = 0$.

Since $x \geq 0$ and $y \geq 0$, $x + y = 0 \Rightarrow x = 0$, which does not satisfy f(x) = x.
Therefore we take x − y − 1 = 0, or y = x − 1.

Substituting into $x^2 = 4 + y$, we obtain $x^2 = x + 3$, or
$x^2 - x - 3 = 0$.

Rejecting the negative root, we have $x = \dfrac{1+\sqrt{13}}{2}$

A car travels downhill at 72 mph (miles per hour), on the level at 63 mph, and uphill at only 56 mph. The car takes 4 hours to travel from town A to town B. The return trip takes 4 hours and 40 minutes. Find the distance between the two towns.

Let the total distance travelled downhill, on the level, and uphill, on the outbound journey, be x, y, and z, respectively. The time taken to travel a distance s at speed v is s/v.

Hence, for the outbound journey

$x/72 + y/63 + z/56 = 4$

While for the return journey, which we assume to be along the same roads

$x/56 + y/63 + z/72 = 14/3$

It may at first seem that we have too little information to solve the puzzle. After all, two equations in three unknowns do not have a unique solution. However, we are not asked for the values of x, y, and z, individually; but for the value of x + y + z.

Multiplying both equations by the least common multiple of denominators 56, 63, and 72, we obtain

$7x + 8y + 9z = 4 \cdot 7 \cdot 8 \cdot 9$
$9x + 8y + 7z = (14/3) \cdot 7 \cdot 8 \cdot 9$

Now it is clear that we should add the equations, yielding

$16(x + y + z) = (26/3) \cdot 7 \cdot 8 \cdot 9$

Therefore x + y + z = 273; the distance between the two towns is 273 miles.

A fair coin is tossed n times. What is the probability that no two consecutive heads appear?

Let f(n) be the number of sequences of heads and tails, of length n, in which two consecutive heads do not appear. The total number of possible sequences from n coin tosses is 2^n.
So the probability that no two consecutive heads occur in n coin tosses is $f(n) / 2^n$.

By enumeration, f(1) = 2, since we have {H, T}, and f(2) = 3, from {HT, TH, TT}.

We then derive a recurrence relation for f(n), as follows.

A sequence of n > 2 coin tosses has no consecutive heads if, and only if:

1. It begins with a tail, and is followed by n−1 tosses with no consecutive heads; or
2. It begins with a head, then a tail, and is followed by n−2 tosses with no consecutive heads.

These two possibilities are mutually exclusive, so we have $f(n) = f(n-1) + f(n-2)$.

This is simply the Fibonacci sequence, shifted forward by two terms.

The Fibonacci sequence is defined by the recurrence equation $F_1 = 1$, $F_2 = 1$, $F_k = F_{k-1} + F_{k-2}$, for k > 2.
So $F_3 = 2$ and $F_4 = 3$, and therefore $f(n) = F_{n+2}$.

A closed form formula for the Fibonacci sequence is
$F_n = (Phi^n - phi^n) / \sqrt{5}$,
where $Phi = (1 + \sqrt{5})/2$ and $phi = (1 - \sqrt{5})/2$ are the roots of the quadratic equation $x^2 - x - 1 = 0$.

Therefore the probability that no two consecutive heads appear in n tosses of a coin is
$F_{n+2} / 2^n = (Phi^{n+2} - phi^{n+2}) / 2^n \cdot \sqrt{5}$.