Math 280 Problems for November 6

Pythagoras Level

1. Find conditions on the parameters \(a, b, c,\) and \(d\) so that
\[
f(x, y) = a \sin(x + y) + b \cos(x + y) + c \sin(x - y) + d \cos(x - y)
\]
can be written as \(f(x, y) = g(x)h(y)\).

2. Find positive integers \(n\) and \(a_1, a_2, \ldots, a_n\) such that
\[
a_1 + a_2 + \cdots + a_n = 2012
\]
and the product \(a_1a_2 \cdots a_n\) is as large as possible.

Newton Level

3. If \(m\) and \(n\) are positive integers and \(a < b\), find a formula for
\[
\int_{a}^{b} \frac{(b - x)^{m}}{m!} \frac{(x - a)^{n}}{n!} dx
\]
and use this to evaluate
\[
\int_{0}^{1} (1 - x^2)^n dx.
\]

4. Sum the series
\[
\sum_{i=1}^{\infty} \frac{36i^2 + 1}{(36i^2 - 1)^2}.
\]
Hint: \(\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}\).

Wiles Level

5. Let \(A, B\) and \(C\) be real square matrices of the same size, and suppose that \(A\) is invertible. Prove that if
\[
(A - B)C = BA^{-1},
\]
then \(C(A - B) = A^{-1}B\).

6. Let \(f\) be a real-valued function with \(n + 1\) derivatives at each point of \(\mathbb{R}\). Show that for each pair of real numbers \(a, b, a < b,\) such that
\[
\ln \left( \frac{f(b) + f'(b) + \cdots + f^{(n)}(b)}{f(a) + f'(a) + \cdots + f^{(n)}(a)} \right) = b - a
\]
there is a number \(c\) in the open interval \((a, b)\) for which
\[
f^{(n+1)}(c) = f(c)\]