Math 280 Problems for September 4

Pythagoras Level

#1

[Kansas Math Competition 2008 #3] The matrix initially looks like this:

$$\begin{pmatrix} 2 & -1 & 1 & \cdots & (-1)^n & (-1)^{n-1} \\ -1 & 2 & -1 & \cdots & (-1)^{n-1} & (-1)^n \\ 1 & -1 & 2 & \cdots & (-1)^n & (-1)^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^n & (-1)^{n-1} & (-1)^n & \cdots & 2 & -1 \\ (-1)^{n-1} & (-1)^n & (-1)^{n-1} & \cdots & -1 & 2 \end{pmatrix}.$$

For each even $i > 1$, add the first row to the i^{th} row, and for each odd $i > 1$, subtract the first row from the i^{th} row. The result is the matrix:

$$\begin{pmatrix} 2 & -1 & 1 & \cdots & (-1)^n & (-1)^{n-1} \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^n & 0 & 0 & \cdots & 1 & 0 \\ (-1)^{n-1} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Now, for each even $i > 1$, add the i^{th} row to the first row, and for each odd $i > 1$, subtract the i^{th} row from the first row. The result is the matrix

$$\begin{pmatrix} n+1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^n & 0 & 0 & \cdots & 1 & 0 \\ (-1)^{n-1} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

This has determinant $n + 1$.

#2

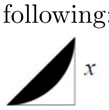
[Kansas Math Competition 2009 #1] The sequence repeats: 1, 3, 2,-1,-3,-2, 1, 3, 2, . . . Each six conseuctive terms sum up to 0. Since 2009 is congruent to 5 mod 6, the first 2009 terms have the same sum as the first 5 terms: 1 + 3 + 2 - 1 - 3 = 2. So the answer is 2.

Newton Level

[Kansas Math Competition 2008 #1] $\lim_{x\to\infty}\left(\frac{x+2a}{x+a}\right)^x = \lim_{x\to\infty}\left(\frac{1+2a/x}{1+a/x}\right)^x = \frac{\lim_{x\to\infty}(1+2a/x)^x}{\lim_{x\to\infty}(1+a/x)^x} = \frac{e^{2a}}{e^a} = e^a =$ 8, so $a = 8$.

#4

[Kansas Math Competition 2008 #2] The figure can be decomposed into infinitely many figures similar to the following:



The total area of the triangle is $x^2/2$. The shaded area A is one-fourth of the difference of a circle of radius x and a square of side length $x\sqrt{2}$, i.e., $A = (\pi x^2 - 2x^2)/4$. Therefore the, shaded area accounts for

$$\frac{(\pi x^2 - 2x^2)/4}{x^2/2} = \frac{\pi - 2}{2}.$$

of the original figure. The total area is 1, so the area of shaded region is $\frac{\pi - 2}{2}$.

Wiles Level

#5

[Kansas Math Competition 2008 #5] Let f_n denote the composition of f with itself n times, i.e.,

$$f_1 = f, \quad f_2 = f \circ f, \quad f_3 = f \circ f \circ f, \ldots$$

Each function f_n is continuously differentiable on $[0,1]$, with $f_n(0) = 0$, $f_n(1) = 1$. By the Mean Value Theorem, there exists $c \in (0,1)$ such that

$$f_n'(c) = \frac{f_n(1) - f_n(0)}{1 - 0} = 1.$$

By Chain Rule,

$$f_n'(c) = 1 = f'(f_{n-1}(c)) \cdot f'(f_{n-2}(c)) \cdots f'(f(c)) \cdot f(c).$$

So define $c_1, \ldots, c_n$ by $c_1 = c$, $c_2 = f(c)$, $c_3 = f(f(c))$, ... , $c_n = f_{n-1}(c)$. Then the third condition provides that the c_i are distinct.

#6

[Kansas Math Competition 2009 #4] To evaluate the integral, make the change of variable $u = bx + a(1-x)$, $du = (b-a)dx$ to get

$$\left(\frac{1}{b-a}\int_a^b u^p du\right)^{\frac{1}{p}} = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} = \exp\left(\frac{1}{p}\ln\left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)\right).$$

Now, to evaluate the limit as $p \to 0$, use L'Hopital's rule:

$$\begin{aligned}
& \exp\lim_{p\to 0}\frac{\ln\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)}{p} \\
= \quad & \exp\lim_{p\to 0}\left(\frac{(p+1)(b-a)}{b^{p+1} - a^{p+1}}\right)\left(\frac{(p+1)(b-a)(b^{p+1}\ln b - a^{p+1}\ln a) - (b^{p+1} - a^{p+1})(b-a)}{(p+1)^2(b-a)^2}\right) \\
= \quad & \exp\lim_{p\to 0}\frac{(p+1)(b^{p+1}\ln b - a^{p+1}\ln a) - (b^{p+1} - a^{p+1})}{(b^{p+1} - a^{p+1})(p+1)} \\
= \quad & \exp\lim_{p\to 0}\left(\frac{b^{p+1}\ln b - a^{p+1}\ln a}{b^{p+1} - a^{p+1}} - 1\right) \\
= \quad & \exp\left(\frac{b\ln b - a\ln a}{b-a} - 1\right) \\
= \quad & e^{-1}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}.
\end{aligned}$$