Math 280 Problems for September 4

Pythagoras Level

#1
[Kansas Math Competition 2008 #3] The matrix initially looks like this:

\[
\begin{pmatrix}
2 & -1 & 1 & \cdots & (-1)^n & (-1)^{n-1} \\
-1 & 2 & -1 & \cdots & (-1)^{n-1} & (-1)^n \\
1 & -1 & 2 & \cdots & (-1)^n & (-1)^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^n & (-1)^{n-1} & (-1)^n & \cdots & 2 & -1 \\
(-1)^{n-1} & (-1)^n & (-1)^{n-1} & \cdots & -1 & 2
\end{pmatrix}
\]

For each even \(i > 1\), add the first row to the \(i\)th row, and for each odd \(i > 1\), subtract the first row from the \(i\)th row. The result is the matrix:

\[
\begin{pmatrix}
2 & -1 & 1 & \cdots & (-1)^n & (-1)^{n-1} \\
-1 & 1 & 0 & \cdots & 0 & 0 \\
1 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^n & 0 & 0 & \cdots & 1 & 0 \\
(-1)^{n-1} & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

Now, for each even \(i > 1\), add the \(i\)th row to the first row, and for each odd \(i > 1\), subtract the \(i\)th row from the first row. The result is the matrix

\[
\begin{pmatrix}
n + 1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 1 & 0 & \cdots & 0 & 0 \\
1 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^n & 0 & 0 & \cdots & 1 & 0 \\
(-1)^{n-1} & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

This has determinant \(n + 1\).

#2
[Kansas Math Competition 2009 #1] The sequence repeats: 1, 3, 2,-1,-3,-2, 1, 3, 2, . . . . Each six consecutive terms sum up to 0. Since 2009 is congruent to 5 mod 6, the first 2009 terms have the same sum as the first 5 terms: \(1 + 3 + 2 - 1 - 3 = 2\). So the answer is 2.

Newton Level

#4
[Kansas Math Competition 2008 #2] The figure can be decomposed into infinitely many figures similar to the following:

The total area of the triangle is \(x^2/2\). The shaded area \(A\) is one-fourth of the difference of a circle of radius \(x\) and a square of side length \(x\sqrt{2}\), i.e., \(A = (\pi x^2 - 2x^2)/4\). Therefore the, shaded area accounts for

\[
\frac{(\pi x^2 - 2x^2)/4}{x^2/2} = \frac{\pi - 2}{2}.
\]

of the original figure. The total area is 1, so the area of shaded region is \(\frac{\pi - 2}{2}\).
Wiles Level

#5
[Kansas Math Competition 2008 #5] Let \( f_n \) denote the composition of \( f \) with itself \( n \) times, i.e.,
\[
 f_1 = f, \quad f_2 = f \circ f, \quad f_3 = f \circ f \circ f, \ldots
\]
Each function \( f_n \) is continuously differentiable on \([0, 1]\), with \( f_n(0) = 0, f_n(1) = 1 \). By the Mean Value Theorem, there exists \( c \in (0, 1) \) such that
\[
 f'_n(c) = \frac{f_n(1) - f_n(0)}{1 - 0} = 1.
\]
By Chain Rule,
\[
 f'_n(c) = 1 = f'(f_{n-1}(c)) \cdot f'(f_{n-2}(c)) \cdots f'(f(c)) \cdot f(c).
\]
So define \( c_1, \ldots, c_n \) by \( c_1 = c, \ c_2 = f(c), \ c_3 = f(f(c)), \ldots, \ c_n = f_{n-1}(c). \) Then the third condition provides that the \( c_i \) are distinct.

#6
[Kansas Math Competition 2009 #4] To evaluate the integral, make the change of variable \( u = bx + a(1-x) \), \( du = (b-a)dx \) to get
\[
 \left( \frac{1}{b-a} \int_a^b u^p du \right)^\frac{1}{p} = \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^\frac{1}{p} = \exp \left( \frac{1}{p} \ln \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right) \right).
\]
Now, to evaluate the limit as \( p \to 0 \), use L’Hopital’s rule:
\[
 \exp \lim_{p \to 0} \frac{\ln \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)}{p} \\
= \exp \lim_{p \to 0} \frac{(p+1)(b-a)\ln b - a^{p+1} \ln a - (b^{p+1} - a^{p+1})(b-a)}{(p+1)^2(b-a)^2} \\
= \exp \lim_{p \to 0} \frac{(b^{p+1} - a^{p+1} \ln a) - (b^{p+1} - a^{p+1})}{(b-a) \frac{b^{p+1} - a^{p+1}}{p+1}} \\
= \exp \left( \frac{b \ln b - a \ln a}{b - a} - 1 \right) \\
= e^{-1} \left( \frac{b^b}{a^a} \right)^{1/(b-a)}.
\]