Math 280 Solutions for September 18

Pythagoras Level

#1. [Iowa MAA 2005 #2] (a) We may write the sum as the sum of two arithmetic progressions:

$$1 + 2 + 3 + \cdots + 2005 = \frac{2005 \cdot 2006}{2}$$

and

$$1 + 2 + 3 + \cdots + 2004 = \frac{2004 \cdot 2005}{2}.$$

Then the sum of the two is

$$\frac{2005(2004 + 2006)}{2} = 2005^2.$$

(b) In general,

$$\begin{aligned}[1 + 2 + 3 + \cdots + (n-1) + n] + [(n-1) + \cdots + 2 + 1] &= \frac{n(n+1)}{2} + \frac{(n-1)n}{2} \\ &= \frac{n^2 + n + n^2 - n}{2} \\ &= n^2.\end{aligned}$$

#2. [Iowa MAA 2005 #4] Player 2 has a winning strategy. Note that the player who moves to square 2003 wins, and the one

who moves to 1996 can move next to 2003. In general the positions from which a win can be assured are numbers which, like 2010, are of the form $7n + 1$. When Player 1 advances the piece k squares from position $7n + 1$ (and this includes position 1), Player 2 moves it $7 - k$ squares to $7n + 7 = 7(n + 1) + 1$. (Note that if the final number were of the form $6n + r$ with $r = 2$; 3; 4 or 5, the first player could win by moving to square r on the first turn, or if $r = 0$, to square 7.)

Newton Level

#3. [Iowa MAA 2005 #3] (a) From $f'(x) = 1 + f(x)$ we get $f''(x) = f'(x)$, and by an easy induction, $f^{(n)}(x) = f'(x)$ for all $n \geq 1$. Thus $f^{(10)}(x) = f'(x) = 1 + f(x)$, and $f^{(10)}(2) = 1 + f(2) = 4$.
(b) From $f'(x) = 1 + f(x)$ we have $\dfrac{dy}{1+y} = dx$, so $\ln(1 + y) = x + C$ for some constant C. Since $y = 3$ when $x = 2$ we have $C = ln4 - 2$, and

$$x - 2 = \ln(1 + y) - ln4 = \ln\left(\frac{1+y}{4}\right).$$

Putting $x = 3$ gives us $\ln\left(\frac{1+y}{4}\right) = 1$, so $(1 + y)/4 = e$, and $y = 4e - 1$.

#4. [Iowa MAA 2005 #8] For $x \leq t \leq 2x$ we have

$$0 < \frac{1}{\sqrt{t^3+4}} \leq \frac{1}{\sqrt{x^3+4}},$$

and therefore

$$\begin{aligned} 0 &< \int_x^{2x} \frac{dt}{\sqrt{t^3+4}} \\ &\leq \int_x^{2x} \frac{dt}{\sqrt{x^3+4}} \\ &= \frac{1}{\sqrt{x^3+4}} \int_x^{2x} dt \\ &= \frac{x}{\sqrt{x^3+4}}. \end{aligned}$$

But

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^3+4}} = 0,$$

so

$$\lim_{x\to\infty} \int_x^{2x} \frac{dt}{\sqrt{t^3+4}} = 0.$$

Wiles Level

#5. [Iowa MAA 2005 #6] It suffices to show that there are infinitely many rational pairs (x, y) with $y^2 - x^2 = a$ and $y > 0$. Thus we want $a = (y-x)(y+x)$. Let b be any nonzero rational number. We will find rational x and y such that $y - x = b$ and $y + x = a/b$. Indeed, the solution of this pair of equations is $x = (a - b^2)/2b$, $y = (a + b^2)/2b$ and both x and y are rational. Moveover, different values of b yield different values of $y - x$, and therefore different pairs (x, y). It remains only to insure that $y > 0$. For this it suffices that $b > 0$ and $b^2 > -a$, and it is clear that there are infinitely many rational values of b satisfying these conditions.

#6. [Putnam 1995 B1] For a given π, no more than three different values of $\pi(x)$ are possible (four would require one part each of size at least 1,2,3,4, and that's already more than 9 elements). If no such x, y exist, each pair $(\pi(x), \pi'(x))$ occurs for at most 1 element of x, and since there are only 3×3 possible pairs, each must occur exactly once. In particular, each value of $\pi(x)$ must occur 3 times. However, clearly any given value of $\pi(x)$ occurs $k\pi(x)$ times, where k is the number of distinct partitions of that size. Thus $\pi(x)$ can occur 3 times only if it equals 1 or 3, but we have three distinct values for which it occurs, contradiction.