Math 280 Problems for September 25

Pythagoras Level

#1.

[Illinois MAA 2009 #1] **Algebraic solution:** By finding a common denominator and clearing fractions, the given condition becomes

$$(x+y+z)(xy+yz+zx)-xyz=0.$$

By expanding this we get

$$(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)+2xyz=0$$

On the other hand,

$$(x+y)(y+z)(z+x)=(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)+2xyz.$$

Thus the given condition is equivalent to $(x+y)(y+z)(z+x)=0$.

Solution using polynomials: Now x, y, and z are the roots of

$$P(t)=(t-x)(t-y)(t-z).$$

However,

$$P(t)=(t-x)(t-y)(t-z)=t^3-(x+y+z)t^2+(xy+yz+zx)t-xyz.$$

Letting $a=x+y+z$, $b=xy+yz+zx$, and using the fact that $(x+y+z)(xy+yz+zx)=xyz$, we have

$$P(t)=t^3-at^2+bt-ab.$$

This can be factored as $P(t)=(t^2+b)(t-a)$. Since $P(x)=P(y)=P(z)=0$, either $a=x$ or $a=y$ or $a=z$. This means either $y+z=0$ or $x+z=0$ or $x+y=0$, as required.

#2.

[Illinois MAA 2009 5] **Solution using the expected value of a random variable:** For $i=1,2,\ldots 7$ let x_i be the random variable which is 1 if the i-th person gets back their own coat, 0 otherwise. Since there are 7 coats, the expected value of x_i is $1/7$. Let $X=x_1+x_2+\cdots+x_7$. The average number of people who get back their own coat is the expected value of X. Since the expected value of the sum of random variables is the sum of the expected values of the random variables, the expected value of X is $7/7=1$.

Solution by counting: Consider the list of all $5040=7!$ possibilities as an array having $7!$ rows and 7 columns. The total number, over all possibilities, of the people who get back their own coat is the number of times that k appears in the k-th column, $k=1,2,\ldots 7$. For each k, there are $6!$ rows in which k occurs in the k-th column, corresponding to the $6!$ permutations of the other 6 numbers. Hence, the total, over all k, is $7\cdot 6!$ or $7!$. Consequently, the average number over all possibilities is $7!/7!=1$.

Newton Level

#3.

[Illinois MAA 2007 #2] The area of this region is 2. The required area is the value of the following integral

$$\int_0^\pi\left(\int_x^\pi g(t)\,dt\right)\,dx.$$

This integral is taken over the region $R=\{(t,x)\mid x\le t\le\pi, 0\le x\le\pi\}$ which is the triangle in the tx-plane bounded by $t=x$, $t=\pi$, and $x=0$. Therefore, the region R can also be described as $R=\{(t,x)\mid 0\le x\le t, 0\le t\le\pi\}$. Hence the desired integral is

$$\int_0^\pi\left(\int_0^t g(t)\,dx\right)\,dt=\int_0^\pi tg(t)\,dt=\int_0^\pi \sin t\,dt=-\cos t\,|_0^\pi=2.$$

#4.

[Illinois MAA 2007 #5] Observe that for every value of k we have that $k^2+k+1=(k+1)^2-(k+1)+1$. Thus,

$$\begin{aligned}\frac{(2^3-1)(3^3-1)(4^3-1)\cdots(n^3-1)}{(2^3+1)(3^3+1)(4^3+1)\cdots(n^3+1)} &= \frac{(2-1)(3-1)\cdots(n-1)(2^2+2+1)(3^2+3+1)\cdots(n^2+n+1)}{(2+1)(3+1)\cdots(n+1)(2^2-2+1)(3^2-3+1)\cdots(n^2-n+1)}\\ &= \frac{1\cdot 2\cdots(n-1)}{3\cdot 4\cdots(n+1)}\cdot\frac{n^2+n+1}{2^2-2+1}\\ &= \frac{2}{n(n+1)}\cdot\frac{n^2+n+1}{3}\\ &= \frac{2n^2+2n+2}{3n^2+3n}.\end{aligned}$$

Thus the desired limit is $\lim_{n\to\infty}\frac{2n^2+2n+2}{3n^2+3n}=\frac{2}{3}$.

Wiles Level

#5.

[Illinois MAA 2008 #4] Let *abcd* be a 4-digit sequence. The sequence

$$1abcd2abcd3abcd4abcd$$

occurs infinitely often since it is the last 20 digits of infinitely many positive integers. Further, none of the digits of these number are erased. When the tape is cut into strips, the cut is either after the first (leftmost) 1, the first a, the first b, or the first c. In these situations, the following 4-digit strips result:

$$\begin{array}{ll} \text{first } 1 & \boxed{\text{abcd}}\ldots \\ \text{first } a & \boxed{\text{bcd2}}\ \boxed{\text{abcd}}\ldots \\ \text{first } b & \boxed{\text{cd2a}}\ \boxed{\text{bcd3}}\ \boxed{\text{abcd}}\ldots \\ \text{first } c & \boxed{\text{d2ab}}\ \boxed{\text{cd3a}}\ \boxed{\text{bcd4}}\ \boxed{\text{abcd}}\ldots \end{array}$$

In any case, the strip $\boxed{\text{abcd}}$ occurs once each time the 20-digit sequence, given above, occurs on the tape. Hence, every 4-digit sequence occurs infinitely often.

#6.

[Putnam 1997 A5] We may discard any solutions for which $a_1\neq a_2$, since those come in pairs; so assume $a_1=a_2$. Similarly, we may assume that $a_3=a_4$, $a_5=a_6$, $a_7=a_8$, $a_9=a_{10}$. Thus we get the equation

$$2/a_1+2/a_3+2/a_5+2/a_7+2/a_9=1.$$

Again, we may assume $a_1=a_3$ and $a_5=a_7$, so we get $4/a_1+4/a_5+2/a_9=1$; and $a_1=a_5$, so $8/a_1+2/a_9=1$. This implies that $(a_1-8)(a_9-2)=16$, which by counting has 5 solutions. Thus N_{10} is odd.