Math 280 Solutions for October 2

Pythagoras Level

1. (Ohio MAA 2006 #4) If

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

and

\[ B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}, \]

then after a direct calculation we get

\[ AB - BA = \begin{pmatrix} bz - cy & ay + bt - bx - dy \\ cx + dz - a\bar{x} - ct & cy - bz \end{pmatrix}. \]

The sum of the diagonal elements for \( AB - BA \) is 0, while the sum of the diagonal elements for the desired matrix is 5. So no such \( A \) and \( B \) exist.

2. (Ohio MAA 2006 #2) Denote the given integers by \( a_1, a_2, a_3...a_n \). Define:

\[ S_1 = a_1 \]

\[ S_2 = a_1 + a_2 \]

\[ \vdots \]

\[ S_n = a_1 + a_2 + \cdots + a_n \]

If one of the numbers \( S_1, S_2, ... S_n \) is a multiple of \( n \) we are done. Otherwise all possible remainders upon division of these numbers by \( n \) are 1, 2, 3, ... \( n - 1 \), i.e., we get more numbers than possible remainders. Therefore, among the numbers \( S_1, S_2, ... S_n \) there are two numbers, say \( S_k \) and \( S_{k+t} \) which give the same remainders upon division by \( n \). We are done because \( S_{k+t} - S_k = a_{k+1} + a_{k+2} + \cdots + a_{k+t} \) is a multiple of \( n \).

Newton Level

3. (Ohio MAA 2006 #5) By using the (known) limit

\[ \lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}, \]

we can write

\[ \lim_{x \to 0} \frac{1 - \cos(1 - \cos x)}{x^2} = \lim_{x \to 0} \frac{1 - \cos(1 - \cos x)}{(1 - \cos x)^2} \left( \frac{1 - \cos x}{x^2} \right)^2 = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}. \]

So we can take \( n = 2 \) and \( a = \frac{1}{8} \).
4. (Putnam 2008 B2) We claim that

\[ F_n(x) = (\ln x - a_n)x^n/n! , \text{ where } a_n = \sum_{k=1}^{n} 1/k. \]

Indeed, temporarily write \( G_n(x) = (\ln x - a_n)x^n/n! \) for \( x > 0 \) and \( n \geq 1 \). Then

\[ \lim_{x \to 0} G_n(x) = 0 \]

and

\[ G'_n(x) = (\ln x - a_n + 1/n)x^{n-1}/(n-1)! = G_{n-1}(x) \]

and the claim follows by the Fundamental Theorem of Calculus and induction on \( n \).

Given the claim, we have \( F_n(1) = -a_n/n! \) and so we need to evaluate

\[ - \lim_{n \to \infty} \frac{a_n}{\ln n} \]

But since the function \( 1/x \) is strictly decreasing for \( x \) positive,

\[ \sum_{k=2}^{n} 1/k = a_n - 1 \]

is bounded below by

\[ \int_{2}^{n} dx/x = \ln n - \ln 2 \]

and above by

\[ \int_{1}^{n} dx/x = \ln n \]

It follows that

\[ \lim_{n \to \infty} \frac{a_n}{\ln n} = 1 \]

and the desired limit is \(-1\).

**Wiles Level**

5. (Ohio MAA 2006 #1) Evaluating at \((0,0)\) gives

\[ f(0+0) = f(0)f(1) + f(0)f(1) \]

so that \( f(1) = 1/2 \) also. Then

\[ f(x) = f(x+0) = f(x)f(1) + f(0)f(1-x) \]

and we find \( f(x) = f(1-x) \). Substituting into the original equation shows: \( f(x+y) = 2f(x)f(y) \). Then

\[ f(x+1) = 2f(x)f(1) = f(x) \]

and we also get

\[ f(-x) = f(1-(-x)) = f(1+x) = f(x). \]

Therefore

\[ f(x-y) = 2f(x)f(-y) = 2f(x)f(y) = f(x+y). \]

Consequently,

\[ f(x) = f \left( \frac{x}{2} + \frac{x}{2} \right) = f \left( \frac{x}{2} - \frac{x}{2} \right) = f(0) = 1/2. \]

6. (Ohio MAA 2006 #3) If \( g \in G \), then \( gA \cap A \neq \emptyset \) since \( A \) has more than half as many elements as \( G \). Therefore, there exist \( a_1 \) and \( a_2 \) such that \( ga_1 = a_2 \) and hence \( g = a_2a_1^{-1} \).