Math 280 Problems for October 9

Pythagoras Level

1. The increasing sequence \( S = \{2, 3, 5, 6, 7, 10, 11, \ldots \} \) consists of all positive integers which are neither a perfect square nor a perfect cube. What is the 500th term of \( S \)?

2. Solve for \( x \) in terms of \( c \):

\[
2 \log_x c - \log_{cx} c - 3 \log_{c^2x} c = 0
\]

Newton Level

3. Alice, Bob, and Carol repeatedly take turns tossing a fair regular six-sided die. Alice begins; Bob always follows Alice; Carol always follows Bob; and Alice always follows Carol. Find the probability that Carol will be the first to toss a six.

4. Find the exact value of

\[
\lim_{x \to 3} \frac{x}{x - 3} \int_{3}^{x} \sin t \, dt.
\]

Wiles Level

5. An object moves 8 cm in a straight line from \( A \) to \( B \), turns at an angle \( \alpha \), measured in radians and chosen at random from the interval \((0, \pi)\), and moves 5 cm in a straight line to \( C \). What is the probability that \( AC < 7 \)?

6. Prove that there exist infinitely many integers \( n \) such that \( n, n + 1, n + 2 \) are each the sum of the squares of two integers. [Example: \( 0 = 0^2 + 0^2 \), \( 1 = 0^2 + 1^2 \), \( 2 = 1^2 + 1^2 \).]