Math 280 Problems for October 23

Pythagoras Level

1. Let $a_1 = 3$ and for $n \geq 1$, $a_{n+1} = a_n^2 - 2$. Prove that if $m \neq n$ then $a_m$ and $a_n$ are relatively prime.

(NCS-MAA 1998 #4) Note first that all $a_n$ are odd (by an easy induction). Assume that $m < n$. Then

\begin{align*}
a_{m+1} &= a_m^2 - 2 \equiv -2 \mod a_m \\
a_{m+2} &= (-2)^2 - 2 \equiv 2 \mod a_m
\end{align*}

And by induction, for every $k \geq 2$, $a_{m+k} \equiv 2 \mod a_m$. Thus $a_n = qa_m + 2$ or $a_n = qa_m - 2$ for some integer $q$, and therefore every common factor of $a_m$ and $a_n$ is a divisor of 2. Since both $a_m$ and $a_n$ are odd, they are relatively prime.

2. Let $f_1(x) = f(x) = \frac{1}{1-x}$, and for $n > 1$, $f_n(x) = f(f_{n-1}(x))$. Evaluate $f_{2011}(2010)$.

(NCS-MAA 1999 #2) We have

\begin{align*}
f_2(x) &= \frac{1}{1 - \frac{1}{1-x}} = \frac{x-1}{x}, \\
f_3(x) &= \frac{1}{1 - \frac{x}{x+1}} = x.
\end{align*}

Then $f_4(x) = f_1(x)$, and for each $n$, $f_n(x) = f_{n+3}(x)$. Since $2011 \equiv 1 \mod 3$, $f_{2011}(x) = f_2(x)$. So $f_{2011}(2010) = \frac{2009}{2010}$.

Note: Composition of functions that are the quotient of two linear functions is equivalent to $2 \times 2$ matrix multiplication. In this case, the fact that $f_3(x) = x$ is related to

\[
\begin{pmatrix}
0 & 1 \\
-1 & 1
\end{pmatrix}^3 = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

Newton Level

3. Find the maximum and minimum values of

\[2x|x| - 5x + 1,\]

for $|x+1| \leq 3$. Justify your answer.

(NCS-MAA 1997 #4) The maximum value is $33/8 = 4.125$, at $x = -5/4$, and the minimum value is $-11$, at $x = -4$. To see this, let

\[f(x) = 2x|x| - 5x + 1 = \begin{cases} 2x^2 - 5x + 1, & x \geq 0 \\ -2x^2 - 5x + 1, & x < 0 \end{cases}\]

Then

\[f'(x) = \begin{cases} 4x - 5, & x \geq 0 \\ -4x - 5, & x < 0 \end{cases}\]

The range $|x+1| \leq 3$ is equivalent to $-3 \leq x+1 \leq 3$; i.e., $-4 \leq x \leq 2$. Candidates for local extrema are at $x = -4$, $-5/4$, $5/4$ and 2, where $f(x)$ has values $-11$, $33/8 = 4.125$, $-17/8 = -2.125$ and $-1$, respectively. Thus the maximum value is $33/8$, at $x = -5/4$, and the minimum is $-11$, at $x = -4$. 

4. Evaluate \[ \int_1^2 \frac{1}{|x^2|} \, dx, \]
where as usual \([u]\) denotes the greatest integer less than or equal to \(u\).

(NCS-MAA 1999 #4) From the definition of the floor function,

\[
\frac{1}{|x^2|} = \begin{cases} 
\frac{1}{2} & 1 \leq x \leq \sqrt{2} \\
\frac{3}{2} & \sqrt{2} \leq x \leq \sqrt{3} \\
\frac{3}{2} & \sqrt{3} \leq x \leq 2
\end{cases}
\]

Then \[
\int_1^2 \frac{1}{|x^2|} \, dx = 1(\sqrt{2} - 1) + \frac{1}{2}(\sqrt{3} - \sqrt{2}) + \frac{1}{3}(2 - \sqrt{3}) = -\frac{1}{3} + \frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3}.
\]

Wiles Level

5. If \(x = \frac{1 + \sqrt{2010}}{2}\),

what is the value of 

\((4x^3 - 2013x - 2010)^{2015}\)?

Justify your answer.

(NCS-MAA 1997 #5) We know \(4x^2 - 4x + 1 = (2x - 1)^2 = 2010\), so \(4x^2 = 4x + 2009\) and \(4x^3 = 4x^2 + 2009x\). Then

\[
4x^3 - 2013x - 2010 = 4x^2 + 2009x - 2013x - 2010
= 4x^2 - 4x - 2010
= 4x^2 - 4x + 1 - 2011
= 2010 - 2011
= -1.
\]

Thus \((4x^3 - 2013x - 2010)^{2015} = (-1)^{2015} = -1\).

6. Given that \(a\), \(b\) and \(c\) are real numbers with \(a < b\) and \(a < c\), prove that

\[a < \frac{bc - a^2}{b + c - 2a} \leq \min\{b, c\} \]

(NCS-MAA 1999 #6) Since \(a < c\) and \(b - a > 0\) then

\[
a(b - a) < c(b - a)
\]

\[
ab - a^2 < cb - ca
\]

\[
ab + ac - a^2 < bc
\]

\[
ab + ac - 2a^2 < bc - a^2
\]

\[
a < \frac{bc - a^2}{b + c - 2a} \quad \text{since} \quad b + c - 2a > 0
\]

So the first inequality is true.

Note that because of the symmetry in \(b\) and \(c\) in the problem, we may assume without loss of generality that \(b \leq c\). Then since \(a \neq b\)

\[
0 < (a^2 - 2ab + b^2) = (a - b)^2
\]

\[
a^2 < -2ab + b^2
\]

\[
bc - a^2 < bc - 2ab + b^2
\]

\[
\frac{bc - a^2}{b + c - 2a} < b = \min\{b, c\}
\]

So the second inequality is true.