Math 280 Solutions for October 30

Pythagoras Level

1. (ICMC 2009 #1) The area between the lines $P = Q + 1/3$ and $P = Q - 1/3$ that is contained in the square $[0, 2] \times [0, 2]$ is equal to $4 - (5/3)^2 = 11/9$. Divide this number by 4 to get the required probability: 11/36.

2. (ICMC 2009 #4) $z \in \mathbb{R}$ and satisfies the equation $x^3 + 3x - 4 = 0$. Since $x^3 + 3x - 4 = (x - 1)(x^2 + x + 4)$ and the discriminant of $x^2 + x + 4$ is -15, the only real solution of $x^3 + 3x - 4 = 0$ is 1. So, $z = 1$.

Newton Level

3. (Iowa MAA 2006 #3) We look at the square of the distance from (x, x^p) to $(1, 0)$: $f(x) = (1 - x)^2 + x^{2p}$. The derivative of f is $f'(x) = 2(px^{2p-1} + x - 1)$. If $p < 1/2$, then $2p - 1 < 0$ and $f'(x) > 0$ for sufficiently small positive values of x. Therefore, f increases for sufficiently small positive values of x and has a maximum greater than $f(0) = 1$. On the other hand, for $p \geq 1/2$, $f(x) \leq (1 - x) + x = 1$.

4. (Illinois MAA 2006 #6) The difference is zero! The curve $y = (1 - x^{1/p})^q$ passes through the points $(0, 1)$ and $(1, 0)$. Also, the first integral is the area in the first quadrant between this curve and the coordinate axes. Solving for x gives $x = (1 - y^{1/q})^p$. Hence, the area between this curve and the coordinate axes (using a "y" integration) is $\int_0^1 (1 - y^{1/q})^p dy$ Since the area is the same, in either case, the difference gives 0.

Wiles Level

5. (Iowa MAA 2005 #7) A necessary condition is that $0 \leq x^3 < 1$, and therefore $0 \leq x < 1$, because $0 \leq \{u\} < 1$ for all real u. So we restrict attention to x with $0 \leq x < 1$. Then

$$\{(x + 1)^3\} = \{x^3 + 3x^2 + 3x + 1\} = x^3 \Leftrightarrow 3x^2 + 3x = n,$$

where n is an integer and $0 \leq x < 1$. The nonnegative root of this quadratic is

$$x = \frac{-3 + \sqrt{9 + 12n}}{6},$$

and this is real and lies in the interval $[0, 1)$ if and only if $0 \leq n \leq 5$.

6. (VTRMC, 1999 #1) Since the value of $f(x, y)$ is unchanged when we swap x with y,

$$\int_0^1 \int_0^x f(x + y)dydx = \frac{1}{2} \int_0^1 \int_0^1 f(x + y)dydx.$$

Also

$$\int_0^1 f(x + y)dy = \int_x^{1+x} f(z)dz = \int_0^1 f(z)dz$$

because $f(z) = f(1 + z)$ for all z. Since $\int_0^1 f(z)dx = 2010$, we conclude that

$$\int_0^1 \int_0^1 f(x + y)dydx = 2010/2 = 1005.$$