NAME:

Def 1. A number \( r \) is rational if and only if there exists integers \( a \) and \( b \) such that \( r = \frac{a}{b} \).

Def 2. \( 0! = 1 \) and for all integers \( n \geq 1 \),
\[
    n! = n(n-1)(n-2)\cdots 2 \cdot 1.
\]

Def 3. The real number \( e \) is given by
\[
e = \sum_{n=0}^{\infty} \frac{1}{n!}.
\]

Axiom 1. There is no integer \( n \) such that \( 0 < n < 1 \).

Axiom 2. The addition or multiplication of two or more integers results in an integer.

Axiom 3. If \( 0 < n < m \), then \( \frac{1}{m} < \frac{1}{n} \).

Axiom 4. \( \sum_{i=0}^{\infty} a_i - \sum_{i=0}^{k} a_i = \sum_{i=k+1}^{\infty} a_i \).

Axiom 5. If \( a_i > 0 \) for all \( i \), then \( \sum_{i=k}^{\infty} a_i > 0 \) for all \( k \).

Using some or all of the above definitions and axioms (as well as simple rules of algebra and fractions), prove the following theorem:

Theorem 1. If \( r = \frac{a}{b} \) where \( a \) and \( b \in \mathbb{Z} \), then
\[
b! \left( r - \sum_{n=0}^{b} \frac{1}{n!} \right) \in \mathbb{Z}.
\]

Theorem 2. For any integer \( b > 0 \),
\[
b! \left( e - \sum_{n=0}^{b} \frac{1}{n!} \right) > 0.
\]

Theorem 3. For integers \( 0 < b \leq n \),
\[
    \frac{b!}{n!} \leq \frac{1}{(b+1)^{n-b}}.
\]

Theorem 4. For \( b \geq 1 \),
\[
    \sum_{k=b+1}^{\infty} \frac{1}{(b+1)^{k-b}} = \frac{1}{b}.
\]

(Hint: Let \( S = \sum_{k=b+1}^{\infty} \frac{1}{(b+1)^{k-b}} \) and evaluate \( (b+1)S - S \).)
Theorem 1. If \( r = \frac{a}{b} \) where \( a \) and \( b \in \mathbb{Z} \), then
\[
b! \left( r - \sum_{n=0}^{b} \frac{1}{n!} \right) \in \mathbb{Z}.
\]

Theorem 2. For any integer \( b > 0 \),
\[
b! \left( e - \sum_{n=0}^{b} \frac{1}{n!} \right) > 0.
\]

Theorem 3. For integers \( 0 < b \leq n \),
\[
\frac{b!}{n!} \leq \frac{1}{(b+1)^{n-b}}.
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Theorem 4. For \( b \geq 1 \),
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\sum_{k=b+1}^{\infty} \frac{1}{(b+1)^{k-b}} = \frac{1}{b}.
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(Hint: Let \( S = \sum_{k=b+1}^{\infty} \frac{1}{(b+1)^{k-b}} \) and evaluate \( (b+1)S - S \).)

Using the theorems that your group members and you just proved, prove the following theorem.

Theorem 5. The real number \( e \) is irrational.