Math 280 Solutions for September 24  

Pythagoras Level

#1. Find a positive integer the first digit of which is 1, which has the property that if this digit is moved to the end of the number, the number is tripled.

[2005 Ohio MAA CONSTUM #1] Suppose the integer is 

\[ n = 1 \cdot 10^m + d_{m-1} \cdot 10^{m-1} + \cdots + d_1 10 + d_0 \]

Then 

\[ d_{m-1} \cdot 10^m + \cdots + d_1 \cdot 10^2 + d_0 \cdot 10 + 1 = 3(1 \cdot 10^m + d_{m-1} \cdot 10^{m-1} + \cdots + d_1 10 + d_0) \]

Hence 

\[ d_{m-1}(10^m - 3 \cdot 10^{m-1}) + \cdots + d_1(10^2 - 3 \cdot 10) + d_0(10 - 3) = 3 \cdot 10^{m-1} \]

or 

\[ 7d_{m-1} \cdot 10^{m-1} + \cdots + 7d_1 \cdot 10 + 7d_0 = 3 \cdot 10^{m-1} \]

Thus 

\[ n - 10^m = \frac{3 \cdot 10^{m-1}}{7} \]

which implies 

\[ n = \frac{10^m + 1}{7} = \frac{99999 \ldots}{7} \]

The fewest number of 9s you need for \( n \) to be an integer is 6, and in this case we have \( n = 142857 \).

#2. Let \( n \geq 1 \) and define \( A = \{1, 2, \ldots, n\} \). Denote the power set of \( A \) (i.e. the set of all subsets of \( A \)) by \( P(A) \). For each subset \( K \subseteq A \), define the following function:

\[ a(K) = \text{the alternating sum of the members of } K, \text{ starting with the largest element and continuing in decreasing order.} \]

For example, \( a(\{1, 4, 6, 7, 9\}) = 9 - 7 + 6 - 4 + 1 \) Find the following sum (justify your answer)

\[ \sum_{K \in P(A)} a(K) \]

[2005 Ohio MAA CONSTUM #10] Let \( P(A) = B \cup C \) where \( B \) = the subsets containing \( n \) and \( C \) = the subsets not containing \( n \). Then \( |B| = |C| = 2^{n-1} \) and there is a bijection between the elements of \( B \) and \( C \) via

\[ \{a_1, a_2, \ldots, a_j\} \leftrightarrow \{a_1, a_2, \ldots, a_j, n\} \]

The combined alternating sum of the two sets above is \( n \) and thus

\[ \sum_{K \in P(A)} a(K) = n2^{n-1}. \]

#3. The graph of a non-negative, differentiable function \( f \) divides the triangle with vertices \((0, 0), (x, 0), \) and \((x, f(x))\) into two parts having equal areas for each positive value of \( x \). Find an explicit expression for \( f(x) \) if \( f(2010) = 2010 \).

[ISMAA 1998 #6] The area under \( y = f(x) \) is half of the area of the triangle. Therefore,

\[ \int_0^x f(x) \, dx = \frac{1}{2} \left( \frac{1}{2} xf(x) \right). \]

Differentiating this expression by using the Fundamental Theorem of Calculus yields,

\[ f(x) = \frac{1}{4}(f(x) + xf'(x)). \]

This gives the differential equation \( y = \frac{1}{4}(y + y') \) or \( 3y = xy' \). This equation can be solved by separation of variables to give \( y = cx^3 \), for some constant \( c \). Finally, using the initial condition gives \( f(x) = x^3/(2010)^2 \).
#4. Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number $a$ such that

$$f'(\frac{a}{x}) = \frac{x}{f(x)}$$

for all $x > 0$.

[Putnam 2005 B3] The functions are precisely $f(x) = cx^d$ for $c, d > 0$ arbitrary except that we must take $c = 1$ in case $d = 1$. To see this, substitute $a/x$ for $x$ in the given equation:

$$f'(x) = \frac{a}{xf(a/x)}.$$

Differentiate:

$$f''(x) = -\frac{a}{x^2f(a/x)} + \frac{a^2f'(a/x)}{x^3f(a/x)^2}.$$

Now substitute to eliminate evaluations at $a/x$:

$$f''(x) = -\frac{f'(x)}{x} + \frac{f'(x)^2}{f(x)}.$$

Clear denominators:

$$xf(x)f''(x) + f(x)f'(x) = xf'(x)^2.$$

Divide through by $f(x)^2$ and rearrange:

$$0 = \frac{f'(x)}{f(x)} + \frac{xf''(x)}{f(x)} - \frac{xf'(x)^2}{f(x)^2}.$$  

The right side is the derivative of $xf'(x)/f(x)$, so that quantity is constant. That is, for some $d$,

$$\frac{f'(x)}{f(x)} = \frac{d}{x}.$$

Integrating yields $f(x) = cx^d$, as desired.

#5. Consider the numbers $a_2 = 11$, $a_3 = 111$, $a_4 = 1111$, $a_5 = 11111$, ... Show that if $n$ is composite, then so is $a_n$.

[2005 Ohio MAA CONSTUM #6] Suppose $k$ is composite. Let $k = mn$ be a nontrival factorization. Then

$$a_k = a_m \times \sum_{i=0}^{n-1} 10^{mi},$$

so $a_k$ is composite.

#6. Suppose that $a, b \in \mathbb{R}$ with $a < b$. Suppose that $f : (a, b) \to \mathbb{R}$. Suppose that $f$ is increasing and satisfies the property that for all $\lambda \in (0, 1)$ and $x, y \in (a, b)$

$$f(\lambda x + (1 - \lambda)y) \lambda f(x) + (1 - \lambda)f(y)$$

Prove that $f$ is continuous on $(a, b)$.

[ICMC 2009 #6] The condition to which $f$ is subject implies that for $r < s < t$,

$$\frac{f(s) - f(r)}{s - r} \leq \frac{f(t) - f(r)}{t - r} \leq \frac{f(t) - f(s)}{t - s}.$$  

Now let $\epsilon > 0$. Let $x_0 \in (s, t) \subset (a, b)$. Choose $w \in \mathbb{N}$ large enough so that $(x_0 - \epsilon/w, x_0 + \epsilon/w) \subset (s, t)$. Let $m = \frac{f(t) - f(x_0)}{t - x_0}$. Let $k$ be equal to the larger of $w$ or $m$. Finally, let $\delta = \epsilon/k$. If $|x - x_0| < \delta$, the inequality above implies that $|f(x) - f(x_0)| < \epsilon$. 