Math 280 Solutions for September 17

Pythagoras Level

#1. The following expression is a rational number. Which one?

$$4\sqrt{3} + \sqrt{129 - 72\sqrt{3}}.$$

[2010 NJUMC #9] Set the expression equal to r, the rational number. Taking the $4\sqrt{3}$ to the other side of the equation, and then squaring both sides, we get

$$129 - 72\sqrt{3} = r^2 + 48 - 8r\sqrt{3}$$

Now, since the $\sqrt{3}$ is irrational, it follows that $-8r = -72$, and hence $r = 9$.

#2. A teacher decides to hold a canned food drive, and asks each of his 29 students to bring in n cans. When the drive is over, the cans completely fill a certain number of boxes which hold exactly 72 cans each, with 3 cans left over. What is the smallest possible value for n?

[2010 NJUMC #10] The given information tells us that $29n \equiv 3 \mod 72$. By the Chinese Remainder Theorem, we may equivalently solve $5n \equiv 3 \mod 8$ and $2n \equiv 3 \mod 9$ simultaneously. By constructing multiplicative inverses of 5 and 2, or simply by inspection, this leads to $n \equiv 7 \mod 8$ and $n \equiv 6 \mod 9$. Finally, we lift to a solution mod 72 in the standard way to get

$$n \equiv 7(9)(1) + 6(8)(-1) \equiv 15 \mod 72$$

So the answer is $n = 15$, and we can easily check that $15 \cdot 29 = 435 = 6 \cdot 72 + 3$.

Newton Level

#3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies $f(-5) = 8$ and $f(0) = 2$, and is **even**. Define a new function g by

$$g(x) = \begin{cases} f(x) & \text{if } x \le 0 \\ 4 - f(x) & \text{if } x > 0. \end{cases}$$

Compute

$$\int_{-5}^{5} g(x)\ dx.$$

[2010 NJUMC #4] The key is to observe that $h(x) := g(x) - 2$ is now an odd function. Indeed, suppose $x > 0$. Then

$$h(-x) = g(-x) - 2 = f(-x) - 2 = f(x) - 2 = -(2 - f(x)) = -(g(x) - 2) = -h(x)$$

(and $h(0) = 0 = -h(0)$). Since the integral of an odd function from -5 to 5 is 0, we have

$$\int_{-5}^{5} g(x)\ dx = \int_{-5}^{5} (h(x) + 2)dx = \int_{-5}^{5} h(x)dx + \int_{-5}^{5} 2dx = 0 + 20 = 20$$

#4. Compute the sum of the infinite series:

$$\sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} = \frac{1^2}{0!} + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots$$

[2010 NJUMC #12] Begin with the Maclaurin Series for e^x, and then build up to to a generating function for this series as follows.

$$g(x) = [x(xe^x)']' = \frac{1}{0!} + \frac{2^2 x}{1!} + \frac{3^2 x^2}{2!} + \frac{4^2 x^3}{3!} + \cdots$$

We find that $g(x) = (x^2 + 3x + 1)e^x$, and so the series we want is simply $g(1) = 5e$.

Wiles Level

#5. A real-valued sequence is defined recursively by $a_0 = 5$ and $a_{n+1} = 6/(8 - a_n)$ for $n \geq 1$. Determine the limit of this sequence, or explain why the limit does not exist.

[2010 NJUMC #11] The first few terms $(5, 2, 1, 6/7, 21/25, \dots)$ suggest that (a_n) is monotone decreasing, and bounded below. To prove this, we compute the difierence between consecutive terms:

$$a_n - a_{n+1} = \frac{a_n^2 - 8a_n + 6}{a_n - 8} = \frac{[a_n - (4 + \sqrt{10})][a_n - (4 - \sqrt{10})]}{a_n - 8}.$$

From this expression, it is clear that if $4 - \sqrt{10} < a_n < 4 + \sqrt{10}$ (which is less than 8), then $4 - \sqrt{10} < a_{n+1} < a_n$. So because our sequence does begin in the desired interval, it follows that (a_n) is monotone decreasing and bounded below by $4 - \sqrt{10}$. Thus, it converges by the Monotone Convergence Theorem. Let $L = \lim a_n = \lim a_{n+1}$. Then, setting the limits of a_{n+1} and $6/(8 - a_n)$ equal, we must have

$$L = \frac{6}{8 - L} \Rightarrow L^2 - 8L + 6 = 0 \Rightarrow L = 4 \pm \sqrt{10}.$$

Since the limit can *not* be $4 + \sqrt{10}$, it must equal $4 - \sqrt{10}$.

#6. After a lousy April Fool's day with not a single bite, a fisherman is extremely lucky for the remaining 29 days of the month, catching at least one fish each day. When he brags about this fact, and tells the total number of fish caught, a mathematician friend observes that there must have been a continuous stretch of days over which precisely 10 fish were caught. What is the maximum number of fish that the fisherman could have caught for the whole month?

[2010 NJUMC #14] If the total number is 49 or more, it is straightforward to construct scenarios in which the fisherman never catches 10 fish over a consecutive span of days. For example, we have

$$49 = (1 + \cdots + 1) + 11 + (1 + \cdots + 1) + 11 + (1 + \cdots + 1)$$

where he catches one fish per day for 9 days in a row (three times). Conversely, if he caught 48 or fewer, we can show the 10 fish span with a pigeonhole argument. Let a_n be the number of fish caught so far on the nth day of April, and consider the numbers:

$$a_1, a_2, a_3, \dots a_{30}, (a_1 + 10), (a_2 + 10), (a_3 + 10), \dots (a_{30} + 10).$$

Note that (a_n) is a strictly increasing sequence. So here we have 60 numbers which are between 0 and 58 (inclusive). Hence, by the pigeonhole principle, two of the numbers must be the same. But the a_i's are distinct! So this means that $a_j = a_i + 10$ for some $i < j$, and therefore the fisherman has caught exactly 10 fish between days $(i + 1)$ and j (inclusive).