Math 280 Problems for September 24

Pythagoras Level

#1. Find a positive integer the first digit of which is 1, which has the property that if this digit is moved to the end of the number, the number is tripled.

#2. Let $n \geq 1$ and define $A = \{1, 2, \ldots, n\}$. Denote the power set of A (i.e. the set of all subsets of A) by $P(A)$. For each subset $K \subseteq A$, define the following function:
$a(K) =$ the alternating sum of the members of K, starting with the largest element and continuing in decreasing order. For example, $a(\{1, 4, 6, 7, 9\}) = 9 - 7 + 6 - 4 + 1$ Find the following sum (justify your answer)

$$\sum_{K \in P(A)} a(K)$$

Newton Level

#3. The graph of a non-negative, differentiable function f divides the triangle with vertices $(0, 0)$, $(x, 0)$, and $(x, f(x))$ into two parts having equal areas for each positive value of x. Find an explicit expression for $f(x)$ if $f(2010) = 2010$.

#4. Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

Wiles Level

#5. Consider the numbers

$$a_2 = 11, \; a_3 = 111, \; a_4 = 1111, \; a_5 = 11111, \; \ldots$$

Show that if n is composite, then so is a_n.

#6. Suppose that $a, b \in \mathbb{R}$ with $a < b$. Suppose that $f : (a, b) \to \mathbb{R}$. Suppose that f is increasing and satisfies the property that for all $\lambda \in (0, 1)$ and $x, y \in (a, b)$

$$f(\lambda x + (1 - \lambda)y)\lambda f(x) + (1 - \lambda)f(y)$$

Prove that f is continuous on (a, b).